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Initial Value Problems of the Rayleigh-Taylor Instability Type

by

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CONTENTS

ABSTRACT 1

1. INTRODUCTION 1

2. ALTERNATIVE FUNDAMENTAL FORMULATIONS 2

2.1 Stream Function Formulation 2

2.2 Velocity Potential Formulation 4

2.3 DNS Formulation 6

3. SOLUTIONS OF TAYLOR INSTABILITY INITIAL VALUE PROBLEMS FOR INVISCID FLUIDS 7

3.1 Solutions for Two-Fluid, Double Half-Space, Single-Interface Configuration with Constant Acceleration 7

3.1.1 General Result with Surface Tension and an Arbitrary Initial Interface Perturbation 7

3.1.2 An Initial Cosine Perturbation Result 9

3.1.3 An Initial Hump Perturbation Result 9

3.1.4 An Initial Groove Perturbation Result 9

3.2 Solutions for a Fluid Sheet with Constant Acceleration 10

3.2.1 General Result with Surface Tension and Arbitrary Initial Perturbations on the Two Surfaces 10

3.2.2 Result for Initial Cosine Perturbations on the Two Surfaces with the Same Wave Numbers and with a Specified Phase Difference 14

3.3 Solutions for Time-Dependent Accelerations 15

3.3.1 Forms of Time-Dependent Accelerations Considered 15

3.3.2 DNS Formulation Applications to the Double Half-Space Problem with Time-Dependent Accelerations 15

3.3.3 Motions of the Surfaces of a Fluid Sheet Induced by Time-Dependent Accelerations 21

3.3.4 Responses of the Interfaces of a Three-Region Composite Domain to Time-Dependent Accelerations 25

4. SOLUTIONS OF TAYLOR INSTABILITY INITIAL VALUE PROBLEMS FOR VISCOUS FLUIDS 29

4.1 Solutions for a Half-Space Configuration with Constant Acceleration 29

4.1.1 Application of the Stream Function Formulation 29

4.1.2 Application of the DNS Formulation 31

4.1.3 Inversion of the Laplace Transform 33

4.2 Solution for a Viscous Fluid Sheet with Constant Acceleration 34

4.3 Interface Motion between Two Half-Spaces of Incompressible, Viscous Fluids under Constant Acceleration 39

INITIAL VALUE PROBLEMS OF THE RAYLEIGH-TAYLOR INSTABILITY TYPE

by

Roy A. Axford

ABSTRACT

Rayleigh-Taylor instability initial value problems are set up in terms of the velocity potential formulation for inviscid fluids, the stream function formulation for viscous fluids, and the DNS formulation, which is applicable to both inviscid and viscous fluids. Explicit solutions of these initial value problems are obtained for half-spaces, double half-spaces, thick and thin sheets, and stratified media for both time-independent and time-dependent imposed accelerations.

1. INTRODUCTION

This report is the first of a contemplated series that will deal with Rayleigh-Taylor instability phenomena. Our objective is to provide a systematic, expository development of Rayleigh-Taylor instability, together with new analytic and quantitative results.

Emphasis is given to alternative formulations and methods of solution applicable to Rayleigh-Taylor instability initial value problems. We consider the existence of stability or instability and follow the actual time response of interfaces regardless of stability or instability occurrence.

Work pertaining to Rayleigh-Taylor instability phenomena for the 90-yr period from 1883 to 1973 is contained in Refs. 1-45. Additional references are cited in Refs. 1-45 and in NASA Literature Search No. 12616 for the period from 1964 to July 28, 1970.

A direct consideration of the equations of continuity and motion and the introduction of a velocity potential might be described as the standard starting points for the determination of the motions of free surfaces or interfaces between dissimilar media when accelerations perpendicular to the mean positions of these surfaces are imposed. However, the velocity potential formulation is of little value in Rayleigh-Taylor instability initial value problems for viscous fluids.

The last paragraph of Sec. 2 in a paper by Bellman and Pennington[7] states: "Note that in this present Section one cannot satisfy the condition that the velocities be zero when t = 0. Apparently because of the linearization performed, one obtains no motion at all if one attempts to satisfy this condition."

The resolution of this paradox was arrived at independently, first by Carrier and Chang[18] and later by me, by treating the initial value problem for viscous fluids by introducing a stream function formulation in linearized analysis. The stream function formulation for Rayleigh-Taylor instability initial value problems involving viscous fluids is presented *ab initio* in Sec. 2.1 from alternative viewpoints, which uncover its underlying physical interpretation.

In Sec. 2.3 we introduce the divergence Navier-Stokes (DNS) formulation. To my knowledge, this method has not been previously applied to obtain explicit solutions of the initial value problems that arise in the consideration of Rayleigh-Taylor instability phenomena. The DNS formulation is simpler to use for Rayleigh-Taylor instability initial value problems involving viscous fluids than the stream function formulation because only second-order, as contrasted with fourth-order, differential operators are encountered. DNS formulation applications to initial value problems are presented in Secs. 3 and 4, together with the stream function and velocity potential formulations, where explicit solutions are deduced.

Here we report only the Eulerian descriptions of hydrodynamics and the two-dimensional Cartesian geometries.

2. ALTERNATIVE FUNDAMENTAL FORMULATIONS

We shall consider various ways of setting up the field equations found to be useful in the analysis of Taylor instability initial value problems. Boundary conditions are introduced at the point where specific problems are solved.

In the ith fluid region the Navier-Stokes equation for an incompressible fluid with a constant viscosity is

$$\rho_i \frac{\partial \vec{V}_i}{\partial t} + \rho_i\left(\vec{V}_i \cdot \text{grad}\right)\vec{V}_i = \rho_i\vec{\phi} - \text{grad } p_i + \mu_i \nabla^2 \vec{V}_i \quad , \tag{2-1}$$

and the continuity equation is

$$\text{div } \vec{V}_i = 0 \quad . \tag{2-2}$$

These equations hold in an accelerating reference frame that moves toward the Y-axis of the inertial frame with an acceleration g_1. The body force per unit mass $\vec{\phi}$ in Eq. (2-1) is given by

$$\vec{\phi} = -(g + g_1)\hat{j} \quad , \tag{2-3}$$

where g is the acceleration due to gravity ($g > 0$). In Eq. (2-3) the imposed acceleration g_1 is positive when the two reference frames move in the positive Y-direction relative to one another, but is negative for motions directed toward the negative Y-direction. Eqs. (2-1) and (2-3) are valid for both time-independent and time-dependent imposed accelerations.

2.1. Stream Function Formulation

If the region subscript i is dropped, the continuity equation, Eq. (2-2), for two-dimensional Cartesian geometry is

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \quad , \tag{2-4}$$

the x-component of the Navier-Stokes equation is

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \frac{\mu}{\rho}\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) \quad , \tag{2-5}$$

and the y-component of the Navier-Stokes equation is

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial y} - g^* + \frac{\mu}{\rho}\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right) , \tag{2-6}$$

where we have set

$$g^* = g + g_1 \quad . \tag{2-7}$$

Equations (2-4)-(2-6) hold for nonlinear analysis. In linear analysis the equation-of-motion components are taken in the following form:

$$\frac{\partial u}{\partial t} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \frac{\mu}{\rho}\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) \tag{2-8}$$

and

$$\frac{\partial v}{\partial t} = -\frac{1}{\rho}\frac{\partial p}{\partial y} - g^* + \frac{\mu}{\rho}\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right) \quad . \tag{2-9}$$

Obtaining solutions for Eqs. (2-4), (2-8), and (2-9) can be reduced to the computation of a scalar function. The corresponding reduction for the nonlinear case, i.e., for Eqs. (2-4)-(2-6), is carried out in Eqs. (2-20)-(2-34).

Consider a stream function $\frac{\partial \psi}{\partial y}$ such that the x-component of the velocity vector is given by

$$u = \frac{\partial^2 \psi}{\partial y^2} \quad , \tag{2-10}$$

and the y-component of the velocity vector is given as

$$v = -\frac{\partial^2}{\partial x \partial y}\psi \quad . \tag{2-11}$$

The reason for taking the stream function as ψ_y rather than just ψ is apparent in Eq. (2-13). Entering Eqs. (2-10) and (2-11) into Eq. (2-4) shows that the continuity equation is satisfied identically.

In terms of the stream function the y-component of the equation of motion, Eq. (2-9), becomes

$$\frac{\partial p}{\partial y} = \rho \frac{\partial^3 \psi}{\partial t \partial x \partial y} - \rho g^* - \mu\left(\frac{\partial^4 \psi}{\partial x^3 \partial y} + \frac{\partial^4 \psi}{\partial x \partial y^3}\right) . \qquad (2\text{-}12)$$

Integrating this equation with respect to y from y = 0 to y = y produces for the pressure field

$$p(x,y,t) = p(x,o,t) + \rho\left[\frac{\partial^2}{\partial t \partial x}\psi(x,y,t) - \frac{\partial^2}{\partial t \partial x}\psi(x,o,t)\right] - \rho g^* y - \mu\left[\frac{\partial^3 \psi}{\partial x^3}(x,y,t) - \frac{\partial^3 \psi}{\partial x^3}(x,o,t) + \frac{\partial^3 \psi}{\partial x \partial y^2}(x,y,t) - \frac{\partial^3 \psi}{\partial x \partial y^2}(x,o,t)\right] . \qquad (2\text{-}13)$$

The stream function was chosen as ψ_y rather than ψ to get this form for the pressure distribution.

In terms of the stream function the x-component of the equation of motion, Eq. (2-8), becomes

$$\frac{\partial p}{\partial x} = \mu\left(\frac{\partial^4 \psi}{\partial x^2 \partial y^2} + \frac{\partial^4 \psi}{\partial y^4}\right) - \rho \frac{\partial^3 \psi}{\partial t \partial y^2} . \qquad (2\text{-}14)$$

Entering the pressure field from Eq. (2-13) into Eq. (2-14) yields

$$\rho\left[\frac{\partial^3}{\partial t \partial x^2}\psi(x,y,t) + \frac{\partial^3}{\partial t \partial y^2}\psi(x,y,t)\right] = \mu\left[\frac{\partial^4}{\partial x^4}\psi(x,y,t) + 2\frac{\partial^4}{\partial x^2 \partial y^2}\psi(x,y,t) + \frac{\partial^4}{\partial y^4}\psi(x,y,t)\right] - \frac{\partial p}{\partial x}(x,o,t) + \rho \frac{\partial^3}{\partial t \partial x^2}\psi(x,o,t) - \mu\left[\frac{\partial^4}{\partial x^4}\psi(x,o,t) + \frac{\partial^4}{\partial x^2 \partial y^2}\psi(x,o,t)\right] . \qquad (2\text{-}15)$$

If we now require that the scalar function ψ satisfy

$$\frac{\partial^3}{\partial t \partial x^2}\psi(x,y,t) + \frac{\partial^3}{\partial t \partial y^2}\psi(x,y,t) = \frac{\mu}{\rho}\left[\frac{\partial^4}{\partial x^4}\psi(x,y,t) + 2\frac{\partial^4}{\partial x^2 \partial y^2}\psi(x,y,t) + \frac{\partial^4}{\partial y^4}\psi(x,y,t)\right] , \qquad (2\text{-}16)$$

Eq. (2-15) splits into this partial differential equation, together with

$$\frac{\partial p}{\partial x}(x,o,t) = \rho \frac{\partial^3}{\partial t \partial x^2}\psi(x,o,t) - \mu\left[\frac{\partial^4}{\partial x^4}\psi(x,o,t) + \frac{\partial^4}{\partial x^2 \partial y^2}\psi(x,o,t)\right] . \qquad (2\text{-}17)$$

Integrating Eq. (2-17) with respect to x gives

$$p(x,o,t) = p(o,o,t) + \rho \frac{\partial^2}{\partial t \partial x}\psi(x,o,t) - \mu\left[\frac{\partial^3}{\partial x^3}\psi(x,o,t) + \frac{\partial^3}{\partial x \partial y^2}\psi(x,o,t)\right] . \qquad (2\text{-}18)$$

With Eq. (2-18) the pressure distribution of Eq. (2-13) takes the form

$$p(x,y,t) = p(o,o,t) + \rho \frac{\partial^2}{\partial t \partial x}\psi(x,y,t) - \rho g^* y - \mu\left[\frac{\partial^3}{\partial x^3}\psi(x,y,t) + \frac{\partial^3}{\partial x \partial y^2}\psi(x,y,t)\right] . \qquad (2\text{-}19)$$

In this formulation the scalar function ψ is determined as a solution of the fourth-order partial differential equation of Eq. (2-16), subject to appropriate boundary conditions for each specific problem. The velocity vector's x-component is found with Eq. (2-10), its y-component is computed with Eq. (2-11), and the pressure distribution is determined from Eq. (2-19).

These results can also be derived by starting from the vector form of the Navier-Stokes equation. For this second derivation, we shall retain the nonlinear terms to obtain the nonlinear counterpart of the partial differential equation of Eq. (2-16). Because the body force per unit mass given in Eq. (2-3) is derivable from the potential,

$$\Omega = g^* y \quad , \qquad (2\text{-}20)$$

that is,

$$\vec{\phi} = - \text{grad } \Omega \quad , \qquad (2\text{-}21)$$

and because the vector identities

$$(\vec{V} \cdot \text{grad})\, \vec{V} = \text{grad}\left(\frac{\vec{V}\cdot\vec{V}}{2}\right) - \vec{V} \times \text{curl}\, \vec{V} \qquad (2\text{-}22)$$

and

$$\nabla^2\vec{V} = \text{grad div}\, \vec{V} - \text{curl curl}\, \vec{V} \qquad (2\text{-}23)$$

hold, the Navier-Stokes equation for an incompressible fluid can be written as

$$\rho\, \frac{\partial\vec{V}}{\partial t} - \rho\, \vec{V} \times \text{curl}\, \vec{V} = -\rho\, \text{grad}\left(\frac{\vec{V}\cdot\vec{V}}{2}\right) - \rho\, \text{grad}\, \Omega$$

$$- \text{grad}\, p - \mu\, \text{curl curl}\, \vec{V} \quad . \qquad (2\text{-}24)$$

Let the vorticity vector $\vec{\omega}$ be defined by

$$\vec{\omega} = \text{curl}\, \vec{V} \quad , \qquad (2\text{-}25)$$

then Eq. (2-24) becomes

$$\rho\, \frac{\partial\vec{V}}{\partial t} - \rho\, \vec{V}\times\vec{\omega} = -\rho\, \text{grad}\left(\frac{\vec{V}\cdot\vec{V}}{2}\right) - \rho\, \text{grad}\, \Omega$$

$$- \text{grad}\, p - \mu\, \text{curl}\, \vec{\omega} \quad . \qquad (2\text{-}26)$$

Because the curl grad operator acting on a scalar produces zero identically, taking the curl of Eq. (2-26) yields

$$\rho\, \frac{\partial}{\partial t}\, \vec{\omega} - \rho\, \text{curl}\, (\vec{V}\times\vec{\omega}) = -\mu\, \text{curl curl}\, \vec{\omega} \, . \qquad (2\text{-}27)$$

With the vector identities

$$\text{curl}\, (\vec{V}\times\vec{\omega}) = (\vec{\omega}\cdot\text{grad})\, \vec{V} - (\vec{V}\cdot\text{grad})\, \vec{\omega} + \vec{V}\, \text{div}\, \vec{\omega}$$

$$- \vec{\omega}\, \text{div}\, \vec{V} \qquad (2\text{-}28)$$

and

$$\text{curl curl}\, \vec{\omega} = \text{grad div}\, \vec{\omega} - \nabla^2\vec{\omega} \quad , \qquad (2\text{-}29)$$

Eq. (2-27) simplifies to

$$\rho\, \frac{\partial\vec{\omega}}{\partial t} + \rho\, (\vec{V}\cdot\text{grad})\, \vec{\omega} = \mu\, \nabla^2\vec{\omega} + \rho\, (\vec{\omega}\cdot\text{grad})\, \vec{V} \qquad (2\text{-}30)$$

because div curl $\vec{V} \equiv 0$, and because the fluid is assumed to be incompressible.

We now specialize the vorticity equation, Eq. (2-30), to two-dimensional Cartesian coordinates. By representing the velocity vector in terms of the stream function ψ_y, as in Eqs. (2-10) and (2-11), the vorticity vector becomes

$$\vec{\omega} = \text{curl}\, \vec{V} = -\left(\frac{\partial^3\psi}{\partial x^2\partial y} + \frac{\partial^3\psi}{\partial y^3}\right)\hat{k} \quad . \qquad (2\text{-}31)$$

Also,

$$(\vec{\omega}\cdot\text{grad})\, \vec{V} = \left[-\frac{\partial}{\partial y}\, \nabla^2\psi\, \hat{k}\cdot\left(\hat{i}\, \frac{\partial}{\partial x} + \hat{j}\, \frac{\partial}{\partial y}\right)\right]\, \vec{V} \equiv 0 \qquad (2\text{-}32)$$

because $\hat{k} \cdot \hat{i} = 0$ and $\hat{k} \cdot \hat{j} = 0$, and

$$\vec{V}\cdot\text{grad} = u\, \frac{\partial}{\partial x} + v\, \frac{\partial}{\partial y} \quad . \qquad (2\text{-}33)$$

Consequently, the kth component of Eq. (2-30) reduces to

$$\rho\, \frac{\partial^2}{\partial t\partial y}\, \nabla^2\psi + \rho\left(\frac{\partial^2\psi}{\partial y^2}\, \frac{\partial}{\partial x} - \frac{\partial^2\psi}{\partial x\partial y}\, \frac{\partial}{\partial y}\right)\frac{\partial}{\partial y}\, \nabla^2\psi$$

$$= \mu\nabla^2\, \frac{\partial}{\partial y}\nabla^2\psi \quad . \qquad (2\text{-}34)$$

Equation (2-34) is the nonlinear counterpart of Eq. (2-16); however, if the nonlinear terms are dropped, Eq. (2-34) becomes

$$\rho\, \frac{\partial}{\partial t}\, \nabla^2\, \frac{\partial\psi}{\partial y} = \mu\, \nabla^2\nabla^2\, \frac{\partial\psi}{\partial y} \quad . \qquad (2\text{-}35)$$

By integrating this with respect to y and setting the arbitrary function of x and t equal to zero, we find from Eq. (2-35) that

$$\rho\, \frac{\partial}{\partial t}\, \nabla^2\psi = \mu\, \nabla^2\nabla^2\psi \quad , \qquad (2\text{-}36)$$

which is the same as Eq. (2-16).

2.2 Velocity Potential Formulation

The representation of the fluid velocity vector as the gradient of a scalar function, the velocity potential, has been used widely in Taylor instability analysis. However, this approach is confined to irrotational flows of inviscid fluids.

Let ϕ be the velocity potential such that the velocity vector is

$$\vec{V} = - \text{grad}\ \phi \quad . \tag{2-37}$$

The continuity equation for an incompressible fluid then becomes

$$\text{div}\ \vec{V} = \text{div grad}\ \phi = \nabla^2\phi = 0 \quad ; \tag{2-38}$$

that is, conservation of mass requires that the velocity potential satisfy Laplace's equation. Consider the form of the Navier-Stokes equation given in Eq. (2-24), namely,

$$\rho \frac{\partial \vec{V}}{\partial t} - \rho \vec{V} \times \text{curl}\ \vec{V} = - \rho\ \text{grad}\left(\frac{\vec{V}\cdot\vec{V}}{2}\right) - \rho\ \text{grad}\ \Omega - \text{grad}\ p - \mu\ \text{curl curl}\ \vec{V} \quad . \tag{2-24}$$

Because curl grad $\phi \equiv 0$, entering Eq. (2-37) into Eq. (2-24) produces

$$- \rho\ \text{grad} \frac{\partial \phi}{\partial t} = - \rho\ \text{grad}\left(\frac{\vec{V}\cdot\vec{V}}{2}\right) - \rho\ \text{grad}\ \Omega - \text{grad}\ p \quad . \tag{2-39}$$

Integrating this relation gives the pressure distribution in the form

$$p = F(t) + \rho \frac{\partial \phi}{\partial t} - \frac{1}{2} \rho\ \text{grad}\ \phi \cdot \text{grad}\ \phi - \rho\Omega \ , \tag{2-40}$$

where $F(t)$ is an arbitrary function of time. Combining Eqs. (2-20) and (2-40) for two-dimensional Cartesian geometry yields

$$p(x,y,t) = F(t) + \rho \frac{\partial \phi}{\partial t}(x,y,t) - \rho g^* y - \frac{1}{2}\left\{\left[\frac{\partial \phi}{\partial x}(x,y,t)\right]^2 + \left[\frac{\partial \phi}{\partial y}(x,y,t)\right]^2\right\} . \tag{2-41}$$

This form of Bernoulli's equation can be used in nonlinear Taylor instability analysis together with Laplace's equation for the velocity potential. In linear analysis Eq. (2-41) for the pressure field is simplified by neglecting the nonlinear terms; that is, the relation

$$p(x,y,t) = F(t) + \rho \frac{\partial \phi}{\partial t}(x,y,t) - \rho g^* y \tag{2-42}$$

in used in linear analysis.

The pressure distribution can also be obtained by the following arguments for linear analysis of inviscid fluids. The linearized Euler equations of motion are

$$\frac{\partial u}{\partial t} = - \frac{1}{\rho} \frac{\partial p}{\partial x} \tag{2-43}$$

and

$$\frac{\partial v}{\partial t} = - \frac{1}{\rho} \frac{\partial p}{\partial y} - g^* \quad . \tag{2-44}$$

Because

$$v = - \frac{\partial \phi}{\partial y} \quad , \tag{2-45}$$

Eq. (2-44) becomes

$$- \frac{\partial}{\partial y} \frac{\partial \phi}{\partial t} = - \frac{1}{\rho} \frac{\partial p}{\partial y} - g^* \quad . \tag{2-46}$$

Integrating this equation from $y = H$ to $y = y$ gives

$$p(x,y,t) - p(x,H,t) = \rho \frac{\partial \phi}{\partial t}(x,y,t) - \rho \frac{\partial \phi}{\partial t}(x,H,t) - \rho g^*(y - H) \quad . \tag{2-47}$$

The x-component of the Euler equation of motion, Eq. (2-43), now becomes

$$\frac{\partial p}{\partial x}(x,H,t) = \rho \frac{\partial^2 \phi}{\partial t \partial x}(x,H,t) \quad , \tag{2-48}$$

which integrates to

$$p(x,H,t) = p(o,H,t) + \rho \frac{\partial \phi}{\partial t}(x,H,t) - \rho \frac{\partial \phi}{\partial t}(o,H,t) \quad . \tag{2-49}$$

With Eq. (2-49), the pressure field of Eq. (2-47) reduces to

$$p(x,y,t) = p(o,H,t) - \rho \frac{\partial \phi}{\partial t}(o,H,t) + \rho \frac{\partial \phi}{\partial t}(x,y,t) - \rho g^*(y - H) \quad . \tag{2-50}$$

This is the same as Eq. (2-42), with the identification

$$F(t) = p(o,H,t) - \rho \frac{\partial \phi}{\partial t}(o,H,t) + \rho g^*H \quad . \quad (2\text{-}51)$$

2.3. DNS Formulation

The DNS formulation is a recasting of the equations of continuity and motion that results by taking the divergence of the Navier-Stokes equation. This formulation simplifies the solution of linear Taylor instability initial value problems for viscous fluids in that only second-order rather than fourth-order differential operators are encountered, as in the stream function formulation discussed above. The DNS formulation can also be used in the Taylor instability analysis of inviscid fluids.

We introduce a scalar function P related to the pressure by

$$P = p + \rho g^*y \quad . \tag{2-52}$$

The x- and y-components of the Navier-Stokes equation in two-dimensional Cartesian geometry assume in terms of this scalar function the following forms:

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{1}{\rho}\frac{\partial P}{\partial x} + \frac{\mu}{\rho}\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) \tag{2-53}$$

and

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} = -\frac{1}{\rho}\frac{\partial P}{\partial y} + \frac{\mu}{\rho}\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right) . \tag{2-54}$$

Adding the results of operating on Eq. (2-53) with $\frac{\partial}{\partial x}$ and on Eq. (2-54) with $\frac{\partial}{\partial y}$ produces

$$\begin{aligned}\frac{\partial}{\partial t}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) &+ \frac{\partial u}{\partial x}\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\frac{\partial v}{\partial y} + 2\frac{\partial v}{\partial x}\frac{\partial u}{\partial y} \\ &+ u\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) + v\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) \\ &= -\frac{1}{\rho}\left(\frac{\partial^2 P}{\partial x^2} + \frac{\partial^2 P}{\partial y^2}\right) + \frac{\mu}{\rho}\left[\frac{\partial^2}{\partial x^2}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) \right. \\ &\left. + \frac{\partial^2}{\partial y^2}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right)\right] \quad .\end{aligned} \tag{2-55}$$

By taking into account the continuity equation for an incompressible fluid, it follows that Eq. (2-55) reduces to

$$\frac{\partial^2 P}{\partial x^2} + \frac{\partial^2 P}{\partial y^2} + \rho\left(\frac{\partial u}{\partial x}\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\frac{\partial v}{\partial y} + 2\frac{\partial u}{\partial y}\frac{\partial v}{\partial x}\right) = 0 \quad . \tag{2-56}$$

In nonlinear analysis this equation is imagined to be solved for the scalar function P in terms of the derivatives of the components of the velocity vector. The result is then entered into Eqs. (2-53) and (2-54) which, accordingly, become a set of coupled nonlinear partial differential equations for the two components of the velocity vector.

In linear analysis simplification is achieved in the following way. When the nonlinear terms are dropped from Eqs. (2-56), (2-53), and (2-54), we have to consider

$$\frac{\partial^2 P}{\partial x^2} + \frac{\partial^2 P}{\partial y^2} = 0 \quad , \tag{2-57}$$

$$\frac{\partial u}{\partial t} = -\frac{1}{\rho}\frac{\partial P}{\partial x} + \frac{\mu}{\rho}\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) \quad , \tag{2-58}$$

and

$$\frac{\partial v}{\partial t} = -\frac{1}{\rho}\frac{\partial P}{\partial y} + \frac{\mu}{\rho}\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right) . \tag{2-59}$$

Consequently, after determining the scalar function P, by solving Laplace's equation, the two components of the velocity vector can be found by solving the two second-order, time-dependent, inhomogeneous diffusion equations given in Eqs. (2-58) and (2-59). This may be contrasted with the stream function formulation where the scalar function ψ must be determined by solving a fourth-order partial differential equation, namely, Eq. (2-16), and where the components of the velocity vector are found subsequently by differentiation.

When the DNS formulation is applied to Taylor instability initial value problems involving inviscid fluids, irrotational flows are not necessarily assumed as they are in the velocity potential formulation. In linear analysis, with the DNS formulation Laplace's equation for the scalar function, P is solved regardless of whether or not the fluid is taken to be inviscid or viscous. The velocity vector components are then found by integrating Eqs. (2-58) and (2-59) with $\mu = 0$ for inviscid fluids and with $\mu \neq 0$ for viscous fluids. Accordingly, the DNS formulation provides a more direct, unified, systematic basis for solving Taylor instability initial value problems than either the stream function formulation or the velocity potential formulation.

3. SOLUTIONS OF TAYLOR INSTABILITY INITIAL VALUE PROBLEMS FOR INVISCID FLUIDS

3.1. Solutions for Two-Fluid, Double Half-Space, Single-Interface Configuration with Constant Acceleration

3.1.1. General Result with Surface Tension and an Arbitrary Initial Interface Perturbation

In this section we show the effect of surface tension T_s on the time response of the interface between two semi-infinite, inviscid fluids. The initial perturbation of the interface is regarded as arbitrary in shape until specific examples of the general solution are worked out.

In the velocity potential formulation we want to solve

$$\frac{\partial^2\phi_1}{\partial x^2}(x,y,t) + \frac{\partial^2\phi_1}{\partial y^2}(x,y,t) = 0 \tag{3-1}$$

in the upper region, $y > 0$ and $-\infty < x < \infty$, and

$$\frac{\partial^2\phi_2}{\partial x^2}(x,y,t) + \frac{\partial^2\phi_2}{\partial y^2}(x,y,t) = 0 \tag{3-2}$$

in the lower region, $y < 0$ and $-\infty < x < \infty$. In the linear approximation the solutions are subject to the kinematic condition

$$\frac{\partial}{\partial t}\eta(x,t) = -\frac{\partial}{\partial y}\phi_1(x,o,t) \tag{3-3}$$

on the interface and on the boundary conditions of continuity of the y-component of the velocity vector, namely,

$$\frac{\partial\phi_1}{\partial y}(x,o,t) = \frac{\partial\phi_2}{\partial y}(x,o,t) \quad ; \tag{3-4}$$

and of pressure continuity, namely,

$$\rho_2\frac{\partial\phi_2}{\partial t}(x,o,t) - \rho_2 g^*\eta(x,t) = \rho_1\frac{\partial\phi_1}{\partial t}(x,o,t) - \rho_1 g^*\eta(x,t) - T_s\frac{\partial^2\eta}{\partial x^2}(x,t) \quad . \tag{3-5}$$

This initial value, boundary value problem can be solved directly with a multiple integral transform technique. A Fourier cosine transform will be used on the x-coordinate, and the Laplace transform will be taken on the time coordinate for time-independent accelerations.

Let the Fourier cosine transform of the velocity potential and of the interface position be defined by

$$\phi_i(k,y,t) = \int_0^\infty dx\,\cos(kx)\,\phi_i(x,y,t), \quad (i = 1,2) \tag{3-6}$$

and

$$\eta(k,t) = \int_0^\infty dx\,\cos(kx)\,\eta(x,t) \quad , \tag{3-7}$$

respectively.

The Fourier cosine transforms of Eqs. (3-1)-(3-5) are

$$\frac{\partial^2}{\partial y^2}\phi_i(k,y,t) - k^2\phi_i(k,y,t) = 0, \quad (i = 1,2) \quad , \tag{3-8}$$

$$\frac{\partial}{\partial t}\eta(k,t) = -\frac{\partial\phi_1}{\partial y}(k,o,t) \quad , \tag{3-9}$$

$$\frac{\partial\phi_1}{\partial y}(k,o,t) = \frac{\partial\phi_2}{\partial y}(k,o,t) \quad , \tag{3-10}$$

and

$$\rho_2\frac{\partial\phi_2}{\partial t}(k,o,t) - \rho_2 g^*\eta(k,t) = \rho_1\frac{\partial\phi_1}{\partial t}(k,o,t) - \rho_1 g^*\eta(k,t) + T_s k^2\,\eta(k,t) \quad . \tag{3-11}$$

From Eq. (3-8) it follows that

$$\phi_1(k,y,t) = A_1(k,t)\,e^{-ky} \quad , y > 0 \tag{3-12}$$

and

$$\phi_2(k,y,t) = A_2(k,t)\ e^{ky}\ ,\ y < 0\ . \qquad (3\text{-}13)$$

With Eqs. (3-12) and (3-13) the Fourier cosine transforms of the kinematic and boundary conditions become

$$\rho_2 \frac{\partial}{\partial t} A_2(k,t) = \rho_1 \frac{\partial}{\partial t} A_1(k,t) + (\rho_2 - \rho_1)\ g^*\eta(k,t)$$

$$+ T_s k^2\ \eta(k,t)\quad , \qquad (3\text{-}14)$$

$$A_2(k,t) = -\ A_1(k,t)\quad , \qquad (3\text{-}15)$$

and

$$\frac{\partial}{\partial t}\ \eta(k,t) = k\ A_1(k,t)\quad . \qquad (3\text{-}16)$$

By eliminating $A_2(k,t)$ from Eqs. (3-14) and (3-15) we obtain

$$\frac{\partial A_1}{\partial t}\ (k,t) = \left[-\ \frac{(\rho_2-\rho_1)}{\rho_2+\rho_1}\ g^*\ -\ \frac{T_s k^2}{\rho_2+\rho_1}\right]\eta(k,t)\ . \qquad (3\text{-}17)$$

Equations (3-16) and (3-17) comprise a set of two first-order, ordinary differential equations for the two Fourier cosine transforms that appear in them. An equivalent second-order ordinary differential equation for the Fourier cosine transform of the interface perturbation is

$$\frac{\partial^2\eta}{\partial t^2}\ (k,t) = \left[-\ \frac{(\rho_2-\rho_1)}{\rho_2+\rho_1}\ g^*k\ -\ \frac{T_s k^3}{\rho_2+\rho_1}\right]\eta(k,t)\ . \qquad (3\text{-}18)$$

We solve Eqs. (3-16) and (3-17) by introducing the following Laplace transforms

$$\eta(k,s) = \int_0^\infty dt\ e^{-st}\ \eta(k,t) \qquad (3\text{-}19)$$

and

$$A_1(k,s) = \int_0^\infty dt\ e^{-st}\ A_1(k,t)\quad . \qquad (3\text{-}20)$$

The Laplace transforms of Eqs. (3-16) and (3-17) are

$$s\ \eta(k,s) - \eta(k,o) = k\ A_1(k,s) \qquad (3\text{-}21)$$

and

$$s\ A_1(k,s) - A_1(k,o) = \left[-\ \frac{(\rho_2-\rho_1)}{\rho_2+\rho_1}\ g^*\ -\ \frac{T_s k^2}{\rho_2+\rho_1}\right]\eta(k,s), \qquad (3\text{-}22)$$

where $\eta(k,o)$ is the Fourier cosine transform of the initial perturbation on the interface. Upon setting $A_1(k,o) = 0$ for a system at rest initially, we find from Eqs. (3-21) and (3-22) that

$$\eta(k,s) = \frac{s\ \eta(k,o)}{s^2 - \left[-\ \frac{(\rho_2-\rho_1)}{\rho_2+\rho_1}\ g^*k\ -\ \frac{T_s k^3}{\rho_2+\rho_1}\right]}\ . \qquad (3\text{-}23)$$

We define a cutoff wave number k_s by

$$k_s = \sqrt{\frac{(\rho_1-\rho_2)g^*}{T_s}}\quad , \qquad (3\text{-}24)$$

and introduce the quantities

$$\sigma^2(k) = \frac{\rho_1-\rho_2}{\rho_1+\rho_2}\ g^*k\left[1 - \left(\frac{k}{k_s}\right)^2\right],\ \text{for } k < k_s\quad , \qquad (3\text{-}25)$$

and

$$\Sigma^2(k) = \frac{\rho_1-\rho_2}{\rho_1+\rho_2}\ g^*k\left[\left(\frac{k}{k_s}\right)^2 - 1\right],\ \text{for } k > k_s\quad . \qquad (3\text{-}26)$$

The inversion theorem for the Fourier cosine transform,

$$\eta(x,s) = \frac{2}{\pi}\int_0^\infty dk\ \cos(xk)\ \eta(k,s)\quad , \qquad (3\text{-}27)$$

in accordance with Eqs. (3-25) and (3-26), splits into the sum of two integrals; that is, the Laplace transform of the interface displacement can be written as

$$\eta(x,s) = \frac{2}{\pi}\int_0^{k_s} dk\ \cos(xk)\ \eta(k,o)\ \frac{s}{s^2-\sigma^2(k)}$$

$$+\frac{2}{\pi}\int_{k_s}^{\infty} dk\ \cos(xk)\ \eta(k,o)\ \frac{s}{s^2+\Sigma^2(k)} \quad . \quad (3\text{-}28)$$

Inversion of this Laplace transform back into the time domain yields the following expression for the space-time dependence of the interface displacement for a time-independent acceleration,

$$\eta(x,t) = \frac{2}{\pi}\int_0^{k_s} dk\ \eta(k,o)\ \cos(xk)\ \cosh\ [\sigma(k)t]$$

$$+\frac{2}{\pi}\int_{k_s}^{\infty} dk\ \eta(k,o)\ \cos(xk)\ \cos\ [\Sigma(k)t]. \quad (3\text{-}29)$$

This result is valid for an initial interface perturbation that is an arbitrary even function of the x-coordinate.

3.1.2. An Initial Cosine Perturbation Result

For example, if the initial interface perturbation is a cosine distribution, namely,

$$\eta(x,o) = a_o\ \cos(k_o x) \quad , \quad (3\text{-}30)$$

so that

$$\eta(k,o) = a_o \int_0^{\infty} dx\ \cos(xk)\ \cos(xk_o) = a_o\ \frac{\pi}{2}\ \delta(k-k_o) \quad , \quad (3\text{-}31)$$

then Eq. (3-29) becomes

$$\eta(x,t) = \frac{2}{\pi}\int_0^{k_s} dk\ a_o\ \frac{\pi}{2}\ \delta(k-k_o)\ \cos(xk)\ \cosh\ [\sigma(k)t]$$

$$+\frac{2}{\pi}\int_{k_s}^{\infty} dk\ a_o\ \frac{\pi}{2}\ \delta(k-k_o)\ \cos(xk)\ \cos\ [\Sigma(k)t] \quad . \quad (3\text{-}32)$$

If $k_o < k_s$, this simplifies to

$$\eta(x,t) = a_o\ \cos(k_o x)\ \cosh\ [\sigma(k_o)t] \quad , \quad (3\text{-}33)$$

but if $k_o > k_s$, Eq. (3-32) reduces to

$$\eta(x,t) = a_o\ \cos(k_o x)\ \cos\ [\Sigma(k_o)t] \quad . \quad (3\text{-}34)$$

3.1.3. An Initial Hump Perturbation Result

As a second example, consider an initial interface perturbation of the shape

$$\eta(x,o) = \frac{a_o c^2}{x^2+c^2} \quad , \quad (3\text{-}35)$$

the Fourier cosine transform of which is

$$\eta(k,o) - a_o c^2 \int_0^{\infty} dx\ \frac{\cos(kx)}{x^2+c^2} = a_o c\ \frac{\pi}{2}\ e^{-ck} \quad . \quad (3\text{-}36)$$

Here the space-time response of the interface to a constant acceleration is given by

$$\eta(x,t) = a_o c \int_0^{k_s} dk\ e^{-ck}\ \cos(xk)\ \cosh\ [\sigma(k)t]$$

$$+\ a_o c \int_{k_s}^{\infty} dk\ e^{-ck}\ \cos(xk)\ \cos\ [\Sigma(k)t] \quad . \quad (3\text{-}37)$$

3.1.4. An Initial Groove Perturbation Result

As a third example, we consider the space-time response of an initial perturbation that is a v-shaped groove defined by

$$\eta(x,o) = \begin{cases} m(x-x_o),\ \text{if}\ 0 \leq |x| \leq x_o \\ o,\ \text{if}\ |x| > x_o \end{cases} \quad , \quad (3\text{-}38)$$

where $m = y_o/x_o$. The Fourier cosine transform of the initial interface perturbation is

$$\eta(k,o) = \int_0^{x_o} dx\ \cos(kx)\ m(x-x_o)$$

$$= -\frac{m}{k^2}\ [1 - \cos(kx_o)] \quad . \quad (3\text{-}39)$$

Consequently, the space-time response of the interface for a constant acceleration is given by

$$\eta(x,t) = -\frac{2}{\pi} m \int_0^{k_s} dk \left[\frac{1-\cos(kx_o)}{k^2}\right] \cos(xk) \cosh[\sigma(k)t]$$

$$- \frac{2m}{\pi} \int_{k_s}^{\infty} dk \left[\frac{1-\cos(kx_o)}{k^2}\right] \cos(xk) \cos[\Sigma(k)t] \quad . \tag{3-40}$$

In the limit of vanishing surface tension we have

$$\lim_{T \to o} k_s \to \infty \quad , \tag{3-41}$$

that is, the cutoff wave number becomes infinite. In this limit Eq. (3-40) simplifies to

$$\eta(x,t) = -\frac{2}{\pi} m \int_0^{\infty} dk \left[\frac{1-\cos(kx_o)}{k^2}\right] \cos(xk) \cosh\left[\sigma_o(k)t\right] \quad , \tag{3-42}$$

where the definition

$$\sigma_o^2(k) \equiv \frac{\rho_1-\rho_2}{\rho_1+\rho_2} g^* k \tag{3-43}$$

has been used. Let $k = u^2$; then, because

$$1 - \cos(kx_o) = 2 \sin^2 (kx_o/2) \quad , \tag{3-44}$$

the result contained in Eq. (3-42) can be written in the alternative form,

$$\eta(x,t) = -\frac{16m}{\pi} \int_0^{\infty} \frac{du}{u^3} \cosh\left[\sigma_o(u^2)t\right] \cos(xu^2) \sin^2\left(\frac{x_o u^2}{2}\right) \quad . \tag{3-45}$$

3.2. Solutions for a Fluid Sheet with Constant Acceleration

3.2.1. General Result with Surface Tension and Arbitrary Initial Perturbations on the Two Surfaces.

The Taylor instability initial value problem for an inviscid fluid sheet will be solved by means of the velocity potential formulation. We want to solve Laplace's equation for the velocity potential

$$\frac{\partial^2\phi}{\partial x^2}(x,y,t) + \frac{\partial^2\phi}{\partial y^2}(x,y,t) = 0 \tag{3-46}$$

in the region defined by the inequalities $\eta_2(x,t) \leq y \leq H + \eta_1(x,t)$ and $-\infty < x < \infty$, where $\eta_2(x,t)$ is the position of the lower surface, and $\eta_1(x,t)$ is the position of the upper surface. For this problem we write the pressure field in two ways, namely,

$$p(x,y,t) = p_o + \rho \frac{\partial\phi}{\partial t} - \rho g^* y - \frac{1}{2} \operatorname{grad} \phi \cdot \operatorname{grad} \phi \tag{3-47}$$

and

$$p(x,y,t) = p_H + \rho \frac{\partial\phi}{\partial t} - \rho g^*(y - H) - \frac{1}{2} \operatorname{grad} \phi \cdot \operatorname{grad} \phi \ , \tag{3-48}$$

where p_o is the pressure at the mean position of the lower surface, and p_H is the pressure at the mean position of the upper surface. The pressure continuity boundary conditions allowing for surface tension are

$$p(x,H+\eta_1,t) = p_H - \frac{T_s \frac{\partial^2}{\partial x^2}\eta_1(x,t)}{\left\{1+\left[\frac{\partial\eta_1}{\partial x}(x,t)\right]^2\right\}^{3/2}} \qquad (3\text{-}49)$$

at $y = H + \eta_1(x,t)$, and

$$p(x,\eta_2,t) - \frac{T_s \frac{\partial^2}{\partial x^2}\eta_2(x,t)}{\left\{1+\left[\frac{\partial\eta_2}{\partial x}(x,t)\right]^2\right\}^{3/2}} = p_o \qquad (3\text{-}50)$$

at $y = \eta_2(x,t)$. With Eq. (3-48) it follows that Eq. (3-49) becomes

$$\rho\frac{\partial\phi}{\partial t}(x,H+\eta_1,t) - \rho g^*\eta_1(x,t) - \frac{1}{2}\,\text{grad}\,\phi\cdot\text{grad}\,\phi =$$

$$-\frac{T_s \frac{\partial^2}{\partial x^2}\eta_1(x,t)}{\left\{1+\left[\frac{\partial\eta_1}{\partial x}(x,t)\right]^2\right\}^{3/2}}, \qquad (3\text{-}51)$$

and entering Eq. (3-47) into Eq. (3-50) produces

$$\rho\frac{\partial\phi}{\partial t}(x,\eta_2,t) - \rho g^*\eta_2(x,t) - \frac{1}{2}\,\text{grad}\,\phi\cdot\text{grad}\,\phi$$

$$= \frac{T_s \frac{\partial^2}{\partial x^2}\eta_2(x,t)}{\left\{1+\left[\frac{\partial\eta_2}{\partial x}(x,t)\right]^2\right\}^{3/2}}. \qquad (3\text{-}52)$$

On the upper surface the kinematic condition is

$$\frac{D}{Dt}\left[y - H - \eta_1(x,t)\right] = 0 \quad , \qquad (3\text{-}53)$$

where the substantial time derivative operator is given by

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} \quad . \qquad (3\text{-}54)$$

Consequently, Eq. (3-53) reduces to

$$v(x,H+\eta_1,t) = \frac{\partial}{\partial t}\eta_1(x,t)$$

$$+ u(x,H+\eta_1,t)\frac{\partial}{\partial x}\eta_1(x,t) \quad . \qquad (3\text{-}55)$$

Likewise, the kinematic condition on the lower surface is found to be

$$v(x,\eta_2,t) = \frac{\partial}{\partial t}\eta_2(x,t) + u(x,\eta_2,t)\frac{\partial}{\partial x}\eta_2(x,t). \qquad (3\text{-}56)$$

In the linear approximation the pressure boundary conditions in Eqs. (3-51) and (3-52) simplify to

$$\rho\frac{\partial\phi}{\partial t}(x,H,t) - \rho g^*\eta_1(x,t) = -T_s\frac{\partial^2\eta_1}{\partial x^2}(x,t) \qquad (3\text{-}57)$$

and

$$\rho\frac{\partial\phi}{\partial t}(x,o,t) - \rho g^*\eta_2(x,t) = T_s\frac{\partial^2\eta_2}{\partial x^2}(x,t) \quad , \qquad (3\text{-}58)$$

respectively. Also, the linear kinematic conditions from Eqs. (3-55) and (3-56) are

$$\frac{\partial}{\partial t}\eta_1(x,t) = v(x,H,t) = -\frac{\partial\phi}{\partial y}(x,H,t) \qquad (3\text{-}59)$$

and

$$\frac{\partial}{\partial t}\eta_2(x,t) = v(x,o,t) = -\frac{\partial\phi}{\partial y}(x,o,t) \quad , \qquad (3\text{-}60)$$

respectively.

The linear initial value, boundary value problem presented by Eqs. (3-46) and (3-57)-(3-60) can be reduced to a system of four, first-order ordinary differential equations. Let the complex Fourier transforms of the velocity potential and interface displacements be defined by

$$\phi(k,y,t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} dx\, e^{ikx}\phi(x,y,t) \qquad (3\text{-}61)$$

and

$$\eta_i(k,t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} dx\, e^{ikx}\eta_i(x,t), \; (i = 1,2) \quad . \qquad (3\text{-}62)$$

The complex Fourier transforms of Eqs. (3-46) and (3-57)-(3-60) are as follows:

$$\frac{\partial^2}{\partial y^2}\,\phi(k,y,t) - k^2\phi(k,y,t) = 0 \quad , \qquad (3\text{-}63)$$

$$\rho\,\frac{\partial\phi}{\partial t}\,(k,H,t) - \rho g^*\eta_1(k,t) = k^2T_s\eta_1(k,t) \quad , \qquad (3\text{-}64)$$

$$\rho\,\frac{\partial\phi}{\partial t}\,(k,o,t) - \rho g^*\eta_2(k,t) = -\,k^2T_s\eta_2(k,t) \quad , \qquad (3\text{-}65)$$

$$\frac{\partial\eta_1}{\partial t}\,(k,t) = -\,\frac{\partial\phi}{\partial y}\,(k,H,t) \quad , \qquad (3\text{-}66)$$

and

$$\frac{\partial}{\partial t}\,\eta_2(k,t) = -\,\frac{\partial\phi}{\partial y}\,(k,o,t) \quad . \qquad (3\text{-}67)$$

The general solution of Eq. (3-63) is

$$\phi(k,y,t) = A_1(k,t)e^{ky} + A_2(k,t)e^{-ky} \quad . \qquad (3\text{-}68)$$

With this general solution for the complex Fourier transform of the velocity potential, the complex Fourier transforms of the boundary and the kinematic conditions given in Eqs. (3-64)-(3-67) reduce to a system of four first-order, ordinary differential equations for $\eta_1(k,t)$, $\eta_2(k,t)$, $A_1(k,t)$, and $A_2(k,t)$. This system can be written in the form

$$\frac{\partial}{\partial t}\begin{pmatrix}\eta_1(k,t)\\ \eta_2(k,t)\\ B_1(k,t)\\ B_2(k,t)\end{pmatrix} = A\begin{pmatrix}\eta_1(k,t)\\ \eta_2(k,t)\\ B_1(k,t)\\ B_2(k,t)\end{pmatrix} , \qquad (3\text{-}69)$$

where

$$B_i(k,t) = k\,A_i(k,t),\ (i = 1,2) \quad , \qquad (3\text{-}70)$$

and the matrix A is given by

$$A = \begin{bmatrix} 0 & 0 & -e^{kH} & e^{-kH} \\ 0 & 0 & -1 & 1 \\ \dfrac{\sigma_1{}^2}{\Delta} & -e^{-kH}\,\dfrac{\sigma_2{}^2}{\Delta} & 0 & 0 \\ -\,\dfrac{\sigma_1{}^2}{\Delta} & e^{kH}\,\dfrac{\sigma_2{}^2}{\Delta} & 0 & 0 \end{bmatrix} . \qquad (3\text{-}71)$$

In Eq. (3-71) we have introduced the following definitions:

$$\Delta \equiv 2\sinh(kH) = e^{kH} - e^{-kH} \quad , \qquad (3\text{-}72)$$

$$\sigma_1{}^2(k) \equiv kg^*\left(1 + \frac{T_s}{\rho}\,\frac{k^2}{g^*}\right) \quad , \qquad (3\text{-}73)$$

and

$$\sigma_2{}^2(k) \equiv kg^*\left(1 - \frac{T_s}{\rho}\,\frac{k^2}{g^*}\right) \quad . \qquad (3\text{-}74)$$

The general solution of the first-order system of Eq. (3-69) can be determined by the method of Laplace transforms. If we introduce the Laplace transforms defined as

$$\eta_i(k,s) = \int_0^\infty dt\; e^{-st}\eta_i(k,t),\ (i = 1,2) \qquad (3\text{-}75)$$

and

$$B_i(k,s) = \int_0^\infty dt\; e^{-st}B_i(k,t),\ (i = 1,2) \quad , \qquad (3\text{-}76)$$

then the Laplace transform of Eq. (3-69) is

$$(sI-A)\begin{pmatrix}\eta_1(k,s)\\ \eta_2(k,s)\\ B_1(k,s)\\ B_2(k,s)\end{pmatrix} = \begin{pmatrix}\eta_1(k,o)\\ \eta_2(k,o)\\ B_1(k,o)\\ B_2(k,o)\end{pmatrix} , \qquad (3\text{-}77)$$

where I is the unit matrix. The solution of Eq. (3-77) for the complex Fourier-Laplace transforms of the interface displacements yields

$$\eta_1(k,s) = \frac{\begin{vmatrix} \eta_1(k,o) & 0 & e^{kH} & -e^{-kH} \\ \eta_2(k,o) & s & 1 & -1 \\ B_1(k,o) & \frac{e^{-kH}}{\Delta}\sigma_2^2 & s & 0 \\ B_2(k,o) & -\frac{e^{kH}}{\Delta}\sigma_2^2 & 0 & s \end{vmatrix}}{\det (s\ I - A)} \tag{3-78}$$

and

$$\eta_2(k,s) = \frac{\begin{vmatrix} s & \eta_1(k,o) & e^{kH} & -e^{-kH} \\ o & \eta_2(k,o) & 1 & -1 \\ -\frac{\sigma_1^2}{\Delta} & B_1(k,o) & s & o \\ \frac{\sigma_1^2}{\Delta} & B_2(k,o) & o & s \end{vmatrix}}{\det (s\ I - A)} . \tag{3-79}$$

The determinant of the matrix s I - A in Eqs. (3-78) and (3-79) has the value

$$\det (s\ I - A) = s^4 + 2\zeta s^2 - (kg^*)^2 \left[1 - \left(\frac{k}{k_s}\right)^4\right] , \tag{3-80}$$

where we define the cutoff wave number as

$$k_s^2 \equiv \frac{\rho g^*}{T_s} \tag{3-81}$$

and the function ζ as

$$\zeta = \frac{\cosh(kH)}{\sinh(kH)} (kg^*) \left(\frac{k}{k_s}\right)^2 . \tag{3-82}$$

This function is even in the Fourier transform variable k. Also, the polynomial det(s I - A) = 0, obtained with Eq. (3-80), has two real roots and two imaginary roots for s if the wave number is less than the cutoff wave number; that is, if $k < k_s$, and four imaginary roots for s if $k > k_s$. Suppose that $0 < k < k_s$, then the two imaginary roots of this polynomial are $\pm iR_1$ with

$$R_1 = \left\{\zeta + \sqrt{\zeta^2 + (kg^*)^2 \left[1 - \left(\frac{k}{k_s}\right)^4\right]}\right\}^{1/2} , \tag{3-83}$$

and the two real roots are $\pm R_2$ with

$$R_2 = \left\{-\zeta + \sqrt{\zeta^2 + (kg^*)^2 \left[1 - \left(\frac{k}{k_s}\right)^4\right]}\right\}^{1/2} . \tag{3-84}$$

When $k_s < k < \infty$ the four imaginary roots of the polynomial are $\pm iR_3$ and $\pm iR_4$ with

$$R_3 = \left\{\zeta + \sqrt{\zeta^2 - (kg^*)^2 \left[\left(\frac{k}{k_s}\right)^4 - 1\right]}\right\}^{1/2} \tag{3-85}$$

and

$$R_4 = \left\{\zeta - \sqrt{\zeta^2 - (kg^*)^2 \left[\left(\frac{k}{k_s}\right)^4 - 1\right]}\right\}^{1/2} . \tag{3-86}$$

If the fluid sheet is assumed to be at rest initially, so that $B_1(k,o) = 0$ and $B_2(k,o) = 0$, then the complex Fourier-Laplace transform of the space-time response of the upper surface of the fluid sheet that comes out of Eq. (3-78) is found to be

$$\eta_1(k,s) = \frac{s}{\det(s\ I - A)} \left\{\eta_1(k,o) \left[s^2 - \sigma_2^2 \frac{\cosh(kH)}{\sinh(kH)}\right] + \eta_2(k,o) \frac{2\sigma_2^2}{\Delta}\right\} . \tag{3-87}$$

By inverting the Laplace transform of this equation with the Bromwich integral, namely,

$$\eta_1(k,t) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} ds\ e^{ts}\ \eta_1(k,s) \quad , \tag{3-88}$$

the complex Fourier transform of the upper surface time response is obtained. This inversion entails the application of the Cauchy residue theorem and the evaluation of the residues at the four first-order poles, which are the zeros of det(s I - A). If the wave number is such that $0 < k < k_s$, we find that the complex Fourier transform of the response of the upper surface is given by

$$\eta_1(k,t) = \frac{\cos(R_1 t)}{2(R_1^2 - \zeta)}\left\{\eta_1(k,o)\left[R_1^2 + \sigma_2^2 \frac{\cosh(kH)}{\sinh(kH)}\right] - \eta_2(k,o)\frac{2\sigma_2^2}{\Delta}\right\} + \frac{\cosh(R_2 t)}{2(R_2^2 + \zeta)}\left\{\eta_1(k,o)\left[R_2^2 - \sigma_2^2 \frac{\cosh(kH)}{\sinh(kH)}\right] + \eta_2(k,o)\frac{2\sigma_2^2}{\Delta}\right\} . \tag{3-89}$$

However, if the wave number is such that $k_s < k < \infty$, the complex Fourier transform of the response of the upper surface is

$$\eta_1(k,t) = \frac{\cos(R_3 t)}{2(R_3^2 - \zeta)}\left\{\eta_1(k,o)\left[R_3^2 + \sigma_2^2 \frac{\cosh(kH)}{\sinh(kH)}\right] - \eta_2(k,o)\frac{2\sigma_2^2}{\Delta}\right\} + \frac{\cos(R_4 t)}{2(R_y^2 - \zeta)}\left\{\eta_1(k,o)\left[R_4^2 + \sigma_2^2 \frac{\cosh(kH)}{\sinh(kH)}\right] - \eta_2(k,o)\frac{2\sigma_2^2}{\Delta}\right\} . \tag{3-90}$$

Similar results can be found for the complex Fourier transform of the time response of the lower surface of the fluid sheet. The actual space-time response of the interfaces then follows by using the inversion theorem for the complex Fourier transform, namely,

$$\eta_i(x,t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} dk\, e^{-ixk}\, \eta_i(k,t) \quad , \quad (i = 1,2) . \tag{3-91}$$

3.2.2. Results for Initial Cosine Perturbations on the Two Surfaces with the Same Wave Numbers and with a Specified Phase Difference

In the limit of zero surface tension, where $k_s \to \infty$, $\zeta \to 0$, $R_1^2 \to kg^*$, and $R_2^2 \to kg^*$, the complex Fourier transform of the upper surface response as given by Eq. (3-89) simplifies to

$$\eta_1(k,t) = \frac{\eta_1(k,o)}{2\sinh(kH)}\left\{\sinh(kH)\,[\cos(t\sqrt{kg^*}) + \cosh(t\sqrt{kg^*})] + \cosh(kH)\,[\cos(t\sqrt{kg^*}) - \cosh(t\sqrt{kg^*})]\right\} + \frac{\eta_2(k,o)}{2\sinh(kH)}[\cosh(t\sqrt{kg^*}) - \cos(t\sqrt{kg^*})] \quad . \tag{3-92}$$

The corresponding result for the complex Fourier transform of the response of the lower surface is found by inverting the Laplace transform

$$\eta_2(k,s) = \frac{\eta_2(k,o)}{2\sinh(kH)\,[s^4 - (kg^*)^2]}\left\{s^3\, 2\sinh(kH) + kg^*\, 2s\cosh(kH) - 2\, kg^*\, s\eta_1(k,o)\right\}, \tag{3-93}$$

obtained from Eq. (3-79) in the limit of zero surface tension. This inverse Laplace transform is given by

$$\eta_2(k,t) = \frac{\eta_1(k,o)}{2\sinh(kH)}[\cos(t\sqrt{kg^*}) - \cosh(t\sqrt{kg^*})] + \frac{\eta_2(k,o)}{2\sinh(kH)}\left\{\sinh(kH)\,[\cosh(t\sqrt{kg^*}) + \cos(t\sqrt{kg^*})] + \cosh(kH)\,[\cosh(t\sqrt{kg^*}) - \cos(t\sqrt{kg^*})]\right\} . \tag{3-94}$$

If the upper surface has an initial perturbation of the form

$$\eta_1(x,o) = a_1 \cos(k_o x + \varepsilon) \quad , \tag{3-95}$$

whereas the lower surface has an initial perturbation

$$\eta_2(x,o) = a_2 \cos(k_o x) \quad , \tag{3-96}$$

then the complex Fourier transforms of the two initial surface perturbations are given by

$$\eta_1(k,o) = \sqrt{\frac{\pi}{2}}\, a_1 \left[e^{i\varepsilon}\, \delta\left(k + k_o\right) + e^{-i\varepsilon}\, \delta\left(k - k_o\right)\right] \tag{3-97}$$

and

$$\eta_2(k,o) = \sqrt{\frac{\pi}{2}}\, a_2 \left[\delta\left(k + k_o\right) + \delta\left(k - k_o\right)\right] \quad . \tag{3-98}$$

By substituting Eqs. (3-97) and (3-98) into Eqs. (3-92) and (3-94), we obtain the following results for the space-time responses of the upper and lower surfaces of an inviscid fluid sheet to a constant acceleration. The upper surface moves in accordance with

$$\eta_1(x,t) = \frac{a_1 \cos\left(k_o x + \varepsilon\right) \cos\left(t\sqrt{k_o g^*}\right)}{1 - e^{-2k_oH}} - \frac{a_1 \cos\left(k_o x + \varepsilon\right) \cosh\left(t\sqrt{k_o g^*}\right) e^{-2k_oH}}{1 - e^{-2k_oH}} + \frac{a_2 \cos\left(k_o x\right) \left[\cosh\left(t\sqrt{k_o g^*}\right) - \cos\left(t\sqrt{k_o g^*}\right)\right] e^{-k_oH}}{1 - e^{-2k_oH}} \tag{3-99}$$

whereas the lower surface moves as

$$\eta_2(x,t) = \frac{a_2 \cos\left(k_o x\right) \cosh\left(t\sqrt{k_o g^*}\right)}{1 - e^{-2k_oH}} + \frac{a_1 \cos\left(k_o x + \varepsilon\right)}{1 - e^{-2k_oH}} \left[\cos\left(t\sqrt{k_o g^*}\right) - \cosh\left(t\sqrt{k_o g^*}\right)\right] e^{-k_oH} - \frac{a_2 \cos\left(k_o x\right) \cos\left(t\sqrt{k_o g^*}\right) e^{-2k_oH}}{1 - e^{-2k_oH}} \quad . \tag{3-100}$$

The first terms on the right-hand sides of Eqs. (3-99) and (3-100), which are valid in the limit of zero surface tension, are what one would expect on the upper and lower surfaces in the limit of a very thick fluid sheet. The remaining terms in these two equations represent interaction effects between the two surfaces that are significant in affecting the time response of thin fluid sheets.

3.3. Solutions for Time-Dependent Accelerations

3.3.1. Forms of Time-Dependent Accelerations Considered

Here we calculate the space-time response of the interface between two inviscid fluids for specified time-dependent accelerations. For the time interval $0 \leqslant t \leqslant T$ the acceleration will be

$$g^*(t) = g_o \left(1 - \frac{t}{T}\right)^{2r-2} \quad , \tag{3-101}$$

where r is a parameter that varies the shape over the time interval. The corresponding impulse ΔV is given by

$$\Delta V = \int_0^T dt\; g^*(t) = \frac{g_o\, T}{2r - 1} \quad , \tag{3-102}$$

so that in terms of the impulse we have

$$g^*(t) = (2r - 1)\, \frac{\Delta V}{T} \left(1 - \frac{t}{T}\right)^{2r-2} \quad . \tag{3-103}$$

Accordingly, only shape parameters such that $r > 1/2$ are of interest. A constant acceleration corresponds to $r = 1$. If $1/2 < r < 1$, the acceleration increases monotonically over the time interval, and if $r > 1$, the acceleration decreases monotonically. As the shape parameter increases the form of the acceleration becomes more and more peaked.

3.3.2. DNS Formulation Applications to the Double Half-Space Problem with Time-Dependent Accelerations

In the linear approximation the DNS formulation for two half-spaces of inviscid fluids requires that we solve

$$\frac{\partial^2 P_1}{\partial x^2} + \frac{\partial^2 P_1}{\partial y^2} = 0 \quad , \tag{3-104}$$

$$\frac{\partial u_1}{\partial t} = -\frac{1}{\rho_1} \frac{\partial P_1}{\partial x} \quad , \tag{3-105}$$

and

$$\frac{\partial v_1}{\partial t} = -\frac{1}{\rho_1} \frac{\partial P_1}{\partial y} \tag{3-106}$$

in the upper region for which $y > 0$ and $-\infty < x < \infty$, and

$$\frac{\partial^2 P_2}{\partial x^2} + \frac{\partial^2 P_2}{\partial y^2} = 0 \quad , \tag{3-107}$$

$$\frac{\partial u_2}{\partial t} = - \frac{1}{\rho_2} \frac{\partial P_2}{\partial x} \quad , \tag{3-108}$$

and

$$\frac{\partial v_2}{\partial t} = - \frac{1}{\rho_2} \frac{\partial P_2}{\partial y} \tag{3-109}$$

in the lower region for which $y < 0$ and $-\infty < x < \infty$. These solutions are subject to the interface kinematic condition

$$\frac{\partial \eta(x,t)}{\partial t} = v_1(x,o,t) \quad , \tag{3-110}$$

the continuity of the y-component of the velocity vector

$$v_1(x,o,t) = v_2(x,o,t) \quad , \tag{3-111}$$

and, in the absence of surface tension, pressure continuity in the form

$$P_1(x,o,t) - \rho_1 g^*(t) \eta(x,t) = P_2(x,o,t) - \rho_2 g^*(t) \eta(x,t) \quad . \tag{3-112}$$

To solve this initial value, boundary value problem, we introduce the following Fourier cosine transforms

$$P_i(k,y,t) = \int_0^\infty dx \cos(kx)\, P_i(x,y,t), (i = 1,2), \tag{3-113}$$

$$v_i(k,y,t) = \int_0^\infty dx \cos(kx)\, v_i(x,y,t), (i = 1,2), \tag{3-114}$$

and

$$\eta(k,t) = \int_0^\infty dx \cos(kx)\, \eta(x,t) \quad . \tag{3-115}$$

The Fourier cosine transforms of Eqs. (3-104), (3-107), (3-106), (3-109), (3-110), (3-111), and (3-112) are, respectively,

$$\frac{\partial^2}{\partial y^2} P_1(k,y,t) - k^2 P_1(k,y,t) = 0 \quad , \tag{3-116}$$

$$\frac{\partial^2}{\partial y^2} P_2(k,y,t) - k^2 P_2(k,y,t) = 0 \quad , \tag{3-117}$$

$$\frac{\partial v_1}{\partial t} (k,y,t) = - \frac{1}{\rho_1} \frac{\partial P_1}{\partial y} (k,y,t) \quad , \tag{3-118}$$

$$\frac{\partial v_2}{\partial t} (k,y,t) = - \frac{1}{\rho_2} \frac{\partial P_2}{\partial y} (k,y,t) \quad , \tag{3-119}$$

$$\frac{\partial}{\partial t} \eta(k,t) = v_1 (k,o,t) \quad , \tag{3-120}$$

$$v_1(k,o,t) = v_2 (k,o,t) \quad , \tag{3-121}$$

and

$$P_1(k,o,t) = P_2 (k,o,t) + (\rho_1 - \rho_2) g^*(t) \eta(k,t) \quad . \tag{3-122}$$

For $y > 0$ the solution of Eq. (3-116) of interest is

$$P_1(k,y,t) = A_1(k,t)e^{-ky}, \tag{3-123}$$

and for $y < 0$ that of Eq. (3-117) is

$$P_2(k,y,t) = A_2(k,t)e^{ky} \quad . \tag{3-124}$$

Entering Eqs. (3-123) and (3-124) into Eqs. (3-118), (3-119), and (3-122) produces

$$\frac{\partial v_1}{\partial t}(k,y,t) = \frac{k}{\rho_1} A_1(k,t)e^{-ky} \quad , \tag{3-125}$$

$$\frac{\partial v_2}{\partial t}(k,y,t) = -\frac{k}{\rho_2} A_2(k,t)e^{ky} \quad , \tag{3-126}$$

and

$$A_1(k,t) = A_2(k,t) + (\rho_1 - \rho_2)\, g^*(t)\, \eta(k,t) \quad . \tag{3-127}$$

Differentiating Eqs. (3-120) and (3-121) with respect to time gives

$$\frac{\partial^2}{\partial t^2}\eta(k,t) = \frac{\partial v_1}{\partial t}(k,o,t) \tag{3-128}$$

and

$$\frac{\partial v_1}{\partial t}(k,o,t) = \frac{\partial v_2}{\partial t}(k,o,t) \quad . \tag{3-129}$$

With Eqs. (3-125) and (3-126) these last two equations become

$$\frac{\partial^2 \eta}{\partial t^2}(k,t) = \frac{k}{\rho_1} A_1(k,t) \tag{3-130}$$

and

$$A_2(k,t) = -\frac{\rho_2}{\rho_1} A_1(k,t) \quad . \tag{3-131}$$

With Eq. (3-131) it follows from Eq. (3-127) that

$$\frac{A_1(k,t)}{\rho_1} = \frac{\rho_1 - \rho_2}{\rho_1 + \rho_2}\, g^*(t)\, \eta(k,t) \quad , \tag{3-132}$$

and combining Eqs. (3-130) and (3-132) yields

$$\frac{\partial^2}{\partial t^2}\eta(k,t) - \frac{\rho_1 - \rho_2}{\rho_1 + \rho_2}\, kg^*(t)\, \eta(k,t) = 0 \tag{3-133}$$

as the governing equation for the Fourier cosine transform of the interface displacement for an arbitrary time-dependent acceleration when surface tension is negligible. We seek solutions of Eq. (3-133) for the acceleration function of Eq. (3-101), that is, of

$$\frac{\partial^2 \eta}{\partial t^2}(k,t) - \frac{\rho_1 - \rho_2}{\rho_1 + \rho_2}\, kg_o\left(1 - \frac{t}{T}\right)^{2r-2} \eta(k,t) = 0 \quad . \tag{3-134}$$

Let

$$\tau = 1 - \frac{t}{T} \tag{3-135}$$

and

$$B^2 = kg_o \frac{\rho_1 - \rho_2}{\rho_1 + \rho_2} T^2 \quad , \tag{3-136}$$

then Eq. (3-134) becomes

$$\frac{\partial^2}{\partial \tau^2}\eta(k,\tau) - B^2 \tau^{2r-2}\, \eta(k,\tau) = 0 \quad . \tag{3-137}$$

To solve Eq. (3-137) we introduce the new dependent variable $Y(k,\tau)$ by

$$\eta(k,\tau) = \sqrt{\tau}\, Y(k,\tau) \quad , \tag{3-138}$$

so that

$$\tau^2 \frac{\partial^2}{\partial \tau^2} Y(k,\tau) + \tau \frac{\partial}{\partial \tau} Y(k,\tau) - \left(\frac{1}{4} + B^2 \tau^{2r}\right) Y(k,\tau) = 0 \quad . \tag{3-139}$$

With the new independent variable

$$z = \tau^r \quad , \tag{3-140}$$

Eq. (3-139) becomes Bessel's equation, namely,

$$z^2 \frac{\partial^2}{\partial z^2} Y(k,z) + z \frac{\partial}{\partial \tau} Y(k,z)$$

$$- \left(\frac{B^2 z^2}{r^2} + \frac{1}{4r^2}\right) Y(k,z) = 0 \quad . \tag{3-141}$$

Explicit solutions of Eq. (3-141) depend upon whether or not $\rho_1 > \rho_2$ or $\rho_1 < \rho_2$. These two cases will be considered in turn. If $\rho_1 > \rho_2$, let

$$B^2 \to B_+^2 \equiv kg_o \frac{\rho_1 - \rho_2}{\rho_1 + \rho_2} T^2 \quad . \tag{3-142}$$

Here the general solution of Eq. (3-141) is

$$Y(k,z) = c_1 I_{\frac{1}{2r}} \left(B_+ \frac{z}{r}\right) + c_2 I_{-\frac{1}{2r}} \left(B_+ \frac{z}{r}\right) \quad , \tag{3-143}$$

provided that 1/2r is not a positive integer. Consequently, the general solution of Eq. (3-137) is

$$\eta(k,\tau) = \sqrt{\tau} \left[c_1 I_{\frac{1}{2r}} \left(B_+ \frac{\tau^r}{r}\right) + c_2 I_{-\frac{1}{2r}} \left(B_+ \frac{\tau^r}{r}\right)\right] \quad , \tag{3-144}$$

when $\rho_1 > \rho_2$.

However, if $\rho_1 < \rho_2$, let $B^2 \to B_-^2$ where

$$B_-^2 = kg_o \frac{\rho_2 - \rho_1}{\rho_2 + \rho_1} T^2 \quad . \tag{3-145}$$

The general solution of Eq. (3-141) is now taken as

$$Y(k,z) = c_3 J_{\frac{1}{2r}} \left(B_- \frac{z}{r}\right) + c_4 J_{-\frac{1}{2r}} \left(B_- \frac{z}{r}\right) \quad , \tag{3-146}$$

provided that 1/2r is not a positive integer, and the general solution of Eq. (3-137) is

$$\eta(k,\tau) = \sqrt{\tau} \left[c_3 J_{\frac{1}{2r}} \left(B_- \frac{\tau^r}{r}\right) + c_4 J_{-\frac{1}{2r}} \left(B_- \frac{\tau^r}{r}\right)\right] \tag{3-147}$$

when $\rho_1 < \rho_2$.

The arbitrary constants c_1, c_2, c_3, and c_4 in the general solutions in Eqs. (3-144) and (3-147) can be evaluated in terms of the initial conditions, which, in view of Eq. (3-135), occur at $\tau = 1$. If $\eta(k,o)$ is the Fourier cosine transform of the initial interface perturbation, then

$$\eta(k,\ t = 0) = \eta(k,\ \tau = 1) \quad . \tag{3-148}$$

Consequently, if we set R = 1/2r, Eqs. (3-144) and (3-147) evaluated at $\tau = 1$ give

$$c_1 I_R (B_+/r) + c_2 I_{-R} (B_+/r) = \eta(k,o) \tag{3-149}$$

and

$$c_3 J_R(B_-/r) + c_4 J_{-R}(B_-/r) = \eta(k,o) \quad . \tag{3-150}$$

Also, we have

$$\frac{\partial}{\partial t} \eta(k,t) = \frac{\partial \tau}{\partial t} \frac{\partial}{\partial \tau} \eta(k,\tau) = - \frac{1}{T} \frac{\partial}{\partial \tau} \eta(k,\tau) \quad , \tag{3-151}$$

so that

$$\frac{\partial}{\partial \tau} \eta(k,\ \tau = 1) = - T\, \eta_t(k,o) \quad . \tag{3-152}$$

Substituting Eqs. (3-144) and (3-147) into this last equation gives

$$c_1 I_R' (B_+/r) + c_2 I_{-R}' (B_+/r) = - \frac{T}{B_+} \left[\eta_t(k,o) + \frac{\eta(k,o)}{2T}\right] \tag{3-153}$$

and

$$c_3 J_R' (B_-/r) + c_4 J_{-R}' (B_-/r) = - \frac{T}{B_-} \left[\eta_t(k,o) + \frac{\eta(k,o)}{2T}\right] \quad . \tag{3-154}$$

The solution of the algebraic set of Eqs. (3-149) and (3-153) is

$$c_1 = \frac{1}{W_I} \left\{ I'_{-R}\left(\frac{B_+}{r}\right) \eta(k,o) + \frac{T}{B_+} \left[\eta_t(k,o) \right.\right.$$

$$\left.\left. + \frac{\eta(k,o)}{2T}\right] I_{-R}\left(\frac{B_+}{r}\right) \right\} \qquad (3\text{-}155)$$

and

$$c_2 = \frac{1}{W_I} \left\{ -I'_R\left(\frac{B_+}{r}\right) \eta(k,o) - \frac{T}{B_+} \left[\eta_t(k,o) \right.\right.$$

$$\left.\left. + \frac{\eta(k,o)}{2T}\right] I_R\left(\frac{B_+}{r}\right) \right\} , \qquad (3\text{-}156)$$

where the Wronskian W_I is given by

$$W_I = I_R\left(\frac{B_+}{r}\right) I'_{-R}\left(\frac{B_+}{r}\right) - I_{-R}\left(\frac{B_+}{r}\right) I'_R\left(\frac{B_+}{r}\right)$$

$$= -\frac{2}{\pi}\frac{r}{B_+} \sin\left(\frac{\pi}{2r}\right) \quad . \qquad (3\text{-}157)$$

The solution of Eqs. (3-150) and (3-154) is

$$c_3 = \frac{1}{W_J} \left\{ J'_{-R}\left(\frac{B_-}{r}\right) \eta(k,o) + J_{-R}\left(\frac{B_-}{r}\right) \frac{T}{B_-} \left[\eta_t(k,o) \right.\right.$$

$$\left.\left. + \frac{\eta(k,o)}{2T}\right] \right\} \qquad (3\text{-}158)$$

and

$$c_4 = -\frac{1}{W_J} \left\{ J'_R\left(\frac{B_-}{r}\right) \eta(k,o) + J_R\left(\frac{B_-}{r}\right) \frac{T}{B_-} \left[\eta_t(k,o) \right.\right.$$

$$\left.\left. + \frac{\eta(k,o)}{2T}\right] \right\} , \qquad (3\text{-}159)$$

where the Wronskian W_J is

$$W_J = J_R\left(\frac{B_-}{r}\right) J'_{-R}\left(\frac{B_-}{r}\right) - J'_R\left(\frac{B_-}{r}\right) J_{-R}\left(\frac{B_-}{r}\right)$$

$$= -\frac{2r}{\pi B_-} \sin\left(\frac{\pi}{2r}\right) \quad . \qquad (3\text{-}160)$$

Therefore, from Eqs. (3-144), (3-155), and (3-156), the Fourier cosine transform of the interface displacement is given by

$$\eta(k,\tau) = \frac{\sqrt{\tau}}{W_I} \left\{ \eta(k,o) \left[I'_{-R}\left(\frac{B_+}{r}\right) I_R\left(\frac{B_+}{r}\tau^r\right) \right.\right.$$

$$\left. - I'_R\left(\frac{B_+}{r}\right) I_{-R}\left(\frac{B_+}{r}\tau^r\right)\right] + \frac{T}{B_+}\left[\eta_t(k,o)\right.$$

$$\left. + \frac{\eta(k,o)}{2T}\right]\left[I_{-R}\left(\frac{B_+}{r}\right) I_R\left(\frac{B_+}{r}\tau^r\right)\right.$$

$$\left.\left. - I_R\left(\frac{B_+}{r}\right) I_{-R}\left(\frac{B_+}{r}\tau^r\right)\right]\right\} \qquad (3\text{-}161)$$

when $\rho_1 > \rho_2$. However, if $\rho_1 > \rho_2$, then the Fourier cosine transform of the interface displacement is

$$\eta(k,\tau) = \frac{\sqrt{\tau}}{W_J} \left\{ \eta(k,o) \left[J'_{-R}\left(\frac{B_-}{r}\right) J_R\left(\frac{B_-\tau^r}{r}\right) \right.\right.$$

$$\left. - J'_R\left(\frac{B_-}{r}\right) J_{-R}\left(\frac{B_-\tau^r}{r}\right)\right] + \frac{T}{B_-}\left[\eta_t(k,o)\right.$$

$$\left. + \frac{\eta(k,o)}{2T}\right]\left[J_{-R}\left(\frac{B_-}{r}\right) J_R\left(\frac{B_-\tau^r}{r}\right)\right.$$

$$\left.\left. - J_R\left(\frac{B_-}{r}\right) J_{-R}\left(\frac{B_-\tau^r}{r}\right)\right]\right\} \quad . \qquad (3\text{-}162)$$

In Eqs. (3-161) and (3-162) the quantity τ depends upon the time as shown in Eq. (3-135). By applying the inversion theorem for the Fourier cosine transform, the space-time response of the interfaces is found to be

$$\eta(x,t) = \frac{2}{\pi}\int_0^\infty dk \cos(x,k)\ \eta(k,\tau) \quad , \qquad (3\text{-}163)$$

where Eq. (3-161) is used for $\eta(k,\tau)$ if $\rho_1 > \rho_2$, and Eq. (3-162) is used if $\rho_1 < \rho_2$.

Explicit results for the Fourier cosine transform of the interface displacement at the time $t = T$ can be determined by evaluating Eqs. (3-161) and (3-162) at $\tau = 0$. For this evaluation the following limits are required:

$$\lim_{\tau\to 0} \sqrt{\tau}\ I_{\frac{1}{2r}}\left(\frac{B_+}{r}\tau^r\right) = 0 \qquad (3\text{-}164)$$

and

$$\lim_{\tau \to 0} \sqrt{\tau}\, I_{-\frac{1}{2r}}\left(\frac{B_+}{r}\tau^r\right) = \frac{(B_+/2r)^{-1/2r}}{\Gamma(1 - 1/2r)} , \quad (3\text{-}165)$$

where Γ is the gamma function of the indicated argument

$$\lim_{\tau \to 0} \sqrt{\tau}\, J_{\frac{1}{2r}}\left(\frac{B_-}{r}\tau^r\right) = 0 \quad (3\text{-}166)$$

and

$$\lim_{\tau \to 0} \sqrt{\tau}\, J_{-\frac{1}{2r}}\left(\frac{B_-}{r}\tau^r\right) = \frac{\left(B_-/2r\right)^{-1/2r}}{\Gamma(1 - 1/2r)} . \quad (3\text{-}167)$$

These limits are obtained directly by using the series representations of the Bessel functions. Taking into account Eqs. (3-164) and (3-165) we find from Eq. (3-161) that

$$\eta(k,T) = \frac{\pi\left(\frac{B_+}{2r}\right)^{1-\frac{1}{2r}}}{\sin\left(\frac{\pi}{2r}\right)\Gamma\left(1 - \frac{1}{2r}\right)}\left\{\eta(k,o)\, I'_{\frac{1}{2r}}\left(\frac{B_+}{r}\right) + \left[\eta_t(k,o) + \frac{\eta(k,o)}{2T}\right]\frac{T}{B_+}\, I_{\frac{1}{2r}}\left(\frac{B_+}{r}\right)\right\} \quad (3\text{-}168)$$

is the Fourier cosine transform of the interface displacement at $t = T$ if $\rho_1 > \rho_2$. The corresponding result for $\rho_1 < \rho_2$ is

$$\eta(k,T) = \frac{\pi\left(\frac{B_-}{2r}\right)^{1-\frac{1}{2r}}}{\sin\left(\frac{\pi}{2r}\right)\Gamma\left(1 - \frac{1}{2r}\right)}\left\{\eta(k,o)\, J'_{\frac{1}{2r}}\left(\frac{B_-}{r}\right) + \left[\eta_t(k,o) + \frac{\eta(k,o)}{2T}\right]\frac{T}{B_-}\, J_{\frac{1}{2r}}\left(\frac{B_-}{r}\right)\right\} . \quad (3\text{-}169)$$

The classical Taylor theory results for a constant acceleration can be recovered from Eqs. (3-168) and (3-169) as a special case for $r = 1$ because $g^*(t) = g_o$ from Eq. (3-101) if $r = 1$. For example, if $r = 1$, Eq. (3-169) simplifies to

$$\eta(k,T) = \frac{\pi\left(B_-/2\right)^{1/2}}{\Gamma(1/2)}\left\{\eta(k,o)\, J'_{\frac{1}{2}}(B_-) + \left[\eta_t(k,o) + \frac{\eta(k,o)}{2T}\right]\frac{T}{B_-}\, J_{\frac{1}{2}}(B_-)\right\} . \quad (3\text{-}170)$$

However,

$$J_{\frac{1}{2}}(B_-) = \sqrt{2/\pi B_-}\,\sin(B_-) , \quad (3\text{-}171)$$

$$J'_{\frac{1}{2}}(B_-) = \sqrt{\frac{2}{\pi B_-}}\left[\cos(B_-) - \frac{1}{2B_-}\sin(B_-)\right] , \quad (3\text{-}172)$$

and

$$\Gamma\left(\tfrac{1}{2}\right) = \sqrt{\pi} . \quad (3\text{-}173)$$

When these last three equations are entered into Eq. (3-170), the result is

$$\eta(k,T) = \eta(k,o)\cos(B_-) + \frac{T}{B_-}\eta_t(k,o)\sin(B_-) , \quad (3\text{-}174)$$

where $\eta_t(k,o) = 0$ for a system at rest initially. Combining Eqs. (3-163) and (3-174) produces

$$\eta(x,T) = \frac{2}{\pi}\int_0^\infty dk\,\cos(xk)\,\eta(k,o)\cos(B_-) \quad (3\text{-}175)$$

for a system at rest initially. If the initial perturbation is a cosine distribution, entering Eq. (3-31) into Eq. (3-175) gives

$$\eta(x,T) = \frac{2}{\pi}\int_0^\infty dk\,\cos(xk)\, a_o \frac{\pi}{2}\,\delta(k - k_o)\cos(B_-) , \quad (3\text{-}176)$$

that is,

$$\eta(x,T) = a_o \cos(k_o x)\cos[B_-(k_o)] , \quad (3\text{-}177)$$

which, with Eq. (3-145), becomes

$$\eta(x,T) = a_o \cos(k_o x) \cos\left(T\sqrt{k_o g_o \frac{\rho_2 - \rho_1}{\rho_2 + \rho_1}}\right). \quad (3\text{-}178)$$

Accordingly, the classical Taylor theory result is recovered as a special case of the more general results obtained above by means of the DNS formulation solved for time-dependent accelerations of the form given in Eq. (3-101).

3.3.3. Motions of the Surfaces of a Fluid Sheet Induced by Time-Dependent Accelerations

We now consider the Taylor instability initial value problem for an inviscid fluid sheet for time-dependent accelerations of the form given in Eq. (3-101). This problem can be disposed of by first determining a set of coupled, second-order differential equations for the Fourier transforms of the displacements of the upper and lower surfaces of the fluid sheet.

With Eq. (3-68) for the Fourier transform of the velocity potential, Eqs. (3-66) and (3-67) become

$$\frac{\partial}{\partial t}\eta_1(k,t) = k\,[-A_1(k,t)e^{kH} + A_2(k,t)e^{-kH}] \quad (3\text{-}179)$$

and

$$\frac{\partial}{\partial t}\eta_2(k,t) = k[-A_1(k,t) + A_2(k,t)] \quad . \quad (3\text{-}180)$$

Differentiating Eqs. (3-179) and (3-180) with respect to time and solving the results for the time derivative of the A's produces

$$\frac{\partial}{\partial t}A_1(k,t) = \frac{1}{2k\sinh(kH)}\left[-\frac{\partial^2}{\partial t^2}\eta_1(k,t) + e^{-kH}\frac{\partial^2}{\partial t^2}\eta_2(k,t)\right] \quad (3\text{-}181)$$

and

$$\frac{\partial}{\partial t}A_2(k,t) = \frac{1}{2k\sinh(kH)}\left[-\frac{\partial^2}{\partial t^2}\eta_1(k,t) + e^{kH}\frac{\partial^2}{\partial t^2}\eta_2(k,t)\right] \quad . \quad (3\text{-}182)$$

Entering Eq. (3-68) into Eqs. (3-64) and (3-65) gives

$$\frac{\partial}{\partial t}A_1(k,t)e^{kH} + \frac{\partial}{\partial t}A_2(k,t)e^{-kH} - g^*(t)\,\eta_1(k,t) = \frac{k^2}{\rho}T_s\,\eta_1(k,t) \quad (3\text{-}183)$$

and

$$\frac{\partial}{\partial t}A_1(k,t) + \frac{\partial}{\partial t}A_2(k,t) - g^*(t)\,\eta_2(k,t) = -\frac{k^2}{\rho}T_s\,\eta_2(k,t) \quad . \quad (3\text{-}184)$$

Combining Eqs. (3-181)-(3-184) yields

$$-\frac{\cosh(kH)}{\sinh(kH)}\frac{\partial^2}{\partial t^2}\eta_1(k,t) + \frac{1}{\sinh(kH)}\frac{\partial^2}{\partial t^2}\eta_2(k,t) = \left[kg^*(t) + \frac{k^3}{\rho}T_s\right]\eta_1(k,t) \quad (3\text{-}185)$$

and

$$-\frac{1}{\sinh(kH)}\frac{\partial^2}{\partial t^2}\eta_1(k,t) + \frac{\cosh(kH)}{\sinh(kH)}\frac{\partial^2}{\partial t^2}\eta_2(k,t) = \left[kg^*(t) - \frac{k^3}{\rho}T_s\right]\eta_2(k,t) \quad . \quad (3\text{-}186)$$

A more convenient form is obtained by solving Eqs. (3-185) and (3-186) explicitly for the second derivatives. When this is done the following set for the Fourier transforms of the surface displacements is found:

$$\frac{\partial^2}{\partial t^2}\eta_1(k,t) + \frac{\cosh(kH)}{\sinh(kH)}\left[kg^*(t) + \frac{k^3}{\rho}T_s\right]\eta_1(k,t) - \frac{1}{\sinh(kH)}\left[kg^*(t) - \frac{k^3}{\rho}T_s\right]\eta_2(k,t) = 0 \quad (3\text{-}187)$$

and

$$\frac{1}{\sinh(kH)}\left[kg^*(t) + \frac{k^3}{\rho} T_s\right] \eta_1(k,t) + \frac{\partial^2}{\partial t^2}\eta_2(k,t)$$

$$- \frac{\cosh(kH)}{\sinh(kH)}\left[kg^*(t) - \frac{k^3}{\rho} T_s\right] \eta_2(k,t) = 0 \quad . \tag{3-188}$$

Explicit analytic solutions of this set can be determined in terms of Bessel functions for time-dependent acceleration functions of the form given in Eq. (3-101) when the surface tension vanishes. Differential operators that are more complex than the Bessel operator arise if the surface tension is nonvanishing. If the acceleration is sinusoidal in time, Eqs. (3-187) and (3-188) comprise a vector Mathieu equation.

To solve Eqs. (3-187) and (3-188), with the acceleration function of Eq. (3-101), when the surface tension vanishes, we introduce the new independent variable as defined in Eq. (3-135) so that

$$\frac{\partial^2}{\partial\tau^2}\eta_1(k,\tau) + \frac{\cosh(kH)}{\sinh(kH)} kg_o\, T^2\, \tau^{2r-2}\, \eta_1(k,\tau)$$

$$- \frac{kg_o\, T^2\, \tau^{2r-2}}{\sinh(kH)} \eta_2(k,\tau) = 0 \tag{3-189}$$

and

$$\frac{kg_o\, T^2\, \tau^{2r-2}}{\sinh(kH)} + \frac{\partial^2}{\partial\tau^2}\eta_2(k,\tau)$$

$$- \frac{\cosh(kH)}{\sinh(kH)} kg_o\, T^2\, \tau^{2r-2}\, \eta_2(k,\tau) = 0 \quad . \tag{3-190}$$

If we introduce the new dependent variables $Y_i(k,\tau)$ as given by

$$\eta_i(k,\tau) = \sqrt{\tau}\, Y_i(k,\tau) \quad , \quad (i = 1,2) \quad , \tag{3-191}$$

and the new independent variable $z = \tau^r$, then Eqs. (3-189) and (3-190) assume the form

$$z^2 \frac{\partial^2 Y_1}{\partial z^2} + z\frac{\partial Y_1}{\partial z} - \frac{1}{4r^2} Y_1 + \frac{\cosh(kH)}{\sinh(kH)} \frac{B_1^2\, z^2}{r^2} Y_1$$

$$- \frac{B_1^2\, z^2}{r^2 \sinh(kH)} Y_2 = 0 \tag{3-192}$$

and

$$\frac{B_1^2\, z^2}{r^2 \sinh(kH)} Y_1 + z^2 \frac{\partial^2}{\partial z^2} Y_2 + z\frac{\partial}{\partial z} Y_2 - \frac{1}{4r^2} Y_2$$

$$- \frac{\cosh(kH)}{\sinh(kH)} \frac{B_1^2\, z^2}{r^2} Y_2 = 0 \quad , \tag{3-193}$$

where

$$B_1^2 = kg_o\, T^2 \quad . \tag{3-194}$$

Let F_1 and F_2 be two linearly independent solutions of

$$z^2 \frac{\partial^2 F}{\partial z^2} + z\frac{\partial F}{\partial z} - \frac{1}{4r^2} F + \mu^2 z^2 F = 0 \quad , \tag{3-195}$$

and let G_1 and G_2 be two linearly independent solutions of

$$z^2 \frac{\partial^2 G}{\partial z^2} + z\frac{\partial G}{\partial z} - \frac{1}{4r^2} G - \nu^2 z^2 G = 0 \quad . \tag{3-196}$$

Then the general solutions of Eqs. (3-195) and (3-196) are

$$Y_1 = c_1 F_1 + c_2 F_2 + c_3 G_1 + c_4 G_2 \tag{3-197}$$

and

$$Y_2 = S_1\left[c_1 F_1 + c_2 F_2\right] + S_2\left[c_3 G_1 + c_4 G_2\right] , \tag{3-198}$$

where the c_i's for $1 \le i \le 4$ are arbitrary constants, and provided that

$$S_1 = e^{-kH} \quad , \tag{3-199}$$

$$S_2 = e^{kH} \quad , \tag{3-200}$$

and

$$\mu = \nu = B_1/r \quad . \tag{3-201}$$

The basis functions required for Eqs. (3-197) and (3-198) are

$$F_1 = J_{\frac{1}{2r}} \; (\mu z) \quad , \tag{3-202}$$

$$F_2 = J_{-\frac{1}{2r}} \; (\mu z) \quad , \tag{3-203}$$

$$G_1 = I_{\frac{1}{2r}} \; (\nu z) \quad , \tag{3-204}$$

and

$$G_2 = I_{-\frac{1}{2r}} \; (\nu z) \quad , \tag{3-205}$$

where standard notations for Bessel functions were used.

The arbitrary constants in the general solutions of Eqs. (3-197) and (3-198) can be ascertained by satisfying the initial conditions placed on the surface displacements. For a system at rest initially, the initial conditions are

$$\frac{\partial}{\partial \tau} \eta_i(k,\tau) \Big|_{\tau=1} = 0 \quad , \quad (i = 1,2) \quad , \tag{3-206}$$

because $t = 0$ corresponds to $\tau = 1$, and

$$\eta_i(k,\tau) \Big|_{\tau=1} = \eta_i(k,o) \quad , \quad (i = 1,2) \quad , \tag{3-207}$$

where the right-hand side of Eq. (3-207) contains the Fourier transforms of the initial displacements. In terms of $R = 1/2r$, the application of Eq. (3-206) shows that

$$c_2 = - c_1 \frac{B_1 J_R^{'} (B_1/r) + \frac{1}{2} J_R (B_1/r)}{B_1 J_{-R}^{'} (B_1/r) + \frac{1}{2} J_{-R} (B_1/r)} \tag{3-208}$$

and

$$c_4 = - c_3 \frac{B_1 I_R^{'} (B_1/r) + \frac{1}{2} I_R (B_1/r)}{B_1 I_{-R}^{'} (B_1/r) + \frac{1}{2} I_{-R} (B_1/r)} \quad . \tag{3-209}$$

Satisfying the initial conditions in Eq. (3-207) produces the following two results:

$$c_1 = \frac{\left[B_1 J_{-R}^{'} (B_1/r) + \frac{1}{2} J_{-R} (B_1/r)\right]}{B_1 W_{1J} \left(S_2 - S_1\right)} \Big[S_2 \, \eta_1(k,o) - \eta_2(k,o)\Big] \tag{3-210}$$

and

$$c_3 = \frac{\left[B_1 I_{-R}^{'} (B_1/r) + \frac{1}{2} I_{-R} (B_1/r)\right]}{B_1 W_{1I} (S_2 - S_1)} \Big[- S_1 \, \eta_1(k,o) + \eta_2(k,o)\Big] \quad , \tag{3-211}$$

where the two Wronskians are defined by

$$W_{1J} = J_R \left(\frac{B_1}{r}\right) J_{-R}^{'} \left(\frac{B_1}{r}\right) - J_R^{'} \left(\frac{B_1}{r}\right) J_{-R} \left(\frac{B_1}{r}\right) = - \frac{2r}{\pi B_1} \sin \left(\frac{\pi}{2r}\right) \tag{3-212}$$

and

$$W_{1I} = I_R \left(\frac{B_1}{r}\right) I_{-R}^{'} \left(\frac{B_1}{r}\right) - I_{-R} \left(\frac{B_1}{r}\right) I_R^{'} \left(\frac{B_1}{r}\right) = - \frac{2r}{\pi B_1} \sin \left(\frac{\pi}{2r}\right) \quad . \tag{3-213}$$

Consequently, the solutions of Eqs. (3-189) and (3-190) for a fluid sheet that is at rest initially are

$$\frac{\eta_1(k,\tau)}{\sqrt{\tau}} = \frac{\left[S_2\ \eta_1(k,o) - \eta_2(k,o)\right]}{B_1\ W_{1J}\ (S_2 - S_1)} \left\{ B_1 \left[J_R\left(\frac{B_1}{r}\ \tau^r\right) J'_{-R}\left(\frac{B_1}{r}\right) - J_{-R}\left(\frac{B_1}{r}\ \tau^n\right) J'_R\left(\frac{B_1}{r}\right)\right] + \frac{1}{2}\left[J_R\left(\frac{B_1}{r}\ \tau^r\right) J_{-R}\left(\frac{B_1}{r}\right) - J_{-R}\left(\frac{B_1}{r}\ \tau^r\right) J_R\left(\frac{B_1}{r}\right)\right]\right\} + \frac{\left[-S_1\ \eta_1(k,o) + \eta_2(k,o)\right]}{B_1\ W_{1I}\ (S_2 - S_1)} \left\{ B_1 \left[I_R\left(\frac{B_1}{r}\ \tau^r\right) I'_{-R}\left(\frac{B_1}{r}\right) - I_{-R}\left(\frac{B_1}{r}\ \tau^r\right) I'_R\left(\frac{B_1}{r}\right)\right] + \frac{1}{2}\left[I_R\left(\frac{B_1}{r}\ \tau^n\right) I_{-R}\left(\frac{B_1}{r}\right) - I_{-R}\left(\frac{B_1}{r}\ \tau^r\right) I_R\left(\frac{B_1}{r}\right)\right]\right\} \tag{3-214}$$

and

$$\frac{\eta_2(k,\tau)}{\sqrt{\tau}} = \frac{S_1\left[S_2\ \eta_1(k,o) - \eta_2(k,o)\right]}{B_1\ W_{1J}\ (S_2 - S_1)} \left\{ B_1 \left[J_R\left(\frac{B_1}{r}\ \tau^r\right) J'_{-R}\left(\frac{B_1}{r}\right) - J_{-R}\left(\frac{B_1}{r}\ \tau^r\right) J'_R\left(\frac{B_1}{r}\right)\right] + \frac{1}{2}\left[J_R\left(\frac{B_1}{r}\ \tau^r\right) J_{-R}\left(\frac{B_1}{r}\right) - J_{-R}\left(\frac{B_1}{r}\ \tau^r\right) J_R\left(\frac{B_1}{r}\right)\right]\right\} + \frac{S_2\left[-S_1\ \eta_1(k,o) + \eta_2(k,o)\right]}{B_1\ W_{1I}\ (S_2 - S_1)} \left\{ B_1 \left[I_R\left(\frac{B_1}{r}\ \tau^r\right) I'_{-R}\left(\frac{B_1}{r}\right) - I_{-R}\left(\frac{B_1}{r}\ \tau^r\right) I'_R\left(\frac{B_1}{r}\right)\right] + \frac{1}{2}\left[I_R\left(\frac{B_1}{r}\ \tau^r\right) I_{-R}\left(\frac{B_1}{r}\right) - I_{-R}\left(\frac{B_1}{r}\ \tau^r\right) I_R\left(\frac{B_1}{r}\right)\right]\right\} . \tag{3-215}$$

These last two results give the Fourier transforms of the motions of the upper and lower surfaces of a fluid sheet for arbitrarily specified initial surface perturbations. The space-time response of the surface motions is obtained from Eqs. (3-214) and (3-215) with the inversion theorem for the Fourier transform once the Fourier transforms of the initial surface displacements have been calculated.

The values of the Fourier transforms of the displacements of the upper and lower surfaces can be determined at $t = T$ by using the limits given in Eqs. (3-164)-(3-167) and the Wronskians from Eqs. (3-212) and (3-213). Taking the limits of Eqs. (3-214) and (3-215) as $\tau = 0$ yields the following two results:

$$\eta_1(k,T) = \frac{\pi\left(RB_1\right)^{1-R}}{(S_2 - S_1)\ B_1\ \sin(\pi R)\ \Gamma(1-R)} \left\{\left[S_2\ \eta_1(k,o) - \eta_2(k,o)\right]\left[B_1\ J'_R\left(\frac{B_1}{r}\right) + \frac{1}{2}\ J_R\left(\frac{B_1}{r}\right)\right] + \left[-S_1\ \eta_1(k,o) + \eta_2(k,o)\right]\left[B_1\ I'_R\left(\frac{B_1}{r}\right) + \frac{1}{2}\ I_R\left(\frac{B_1}{r}\right)\right]\right\} \tag{3-216}$$

and

$$\eta_2(k,T) = \frac{\pi\left(RB_1\right)^{1-R}}{(S_2 - S_1)\; B_1\; \sin(\pi R)\; \Gamma(1-R)} \left\{ S_1\left[S_2\; \eta_1(k,o) - \eta_2(k,o)\right]\left[B_1\; J_R'\left(\frac{B_1}{r}\right) + \frac{1}{2}\; J_R\left(\frac{B_1}{r}\right)\right] + S_2\left[-S_1\; \eta_1(k,o) + \eta_2(k,o)\right]\left[B_1\; I_R'\left(\frac{B_1}{r}\right) + \frac{1}{2}\; I_R\left(\frac{B_1}{r}\right)\right]\right\} \quad . \tag{3-217}$$

Using the Fourier transform inversion theorem in conjunction with Eqs. (3-216) and (3-217) produces the spatial dependence of the displacements of the upper and lower surfaces at $t = T$ for arbitrary initial surface perturbations.

3.3.4. Responses of the Interfaces of a Three-Region Composite Domain to Time-Dependent Accelerations

A class of solutions of the Taylor instability initial value problem will be obtained for a three-region composite domain with two interfaces. We seek the space-time response of the interfaces for time-dependent accelerations of the form given in Eq. (3-101).

In the velocity potential formulation it is necessary to solve Laplace's equation

$$\frac{\partial^2\phi_i}{\partial x^2} + \frac{\partial^2\phi_i}{\partial y^2} = 0 \quad , \quad (i = 1,\, 2,\, 3) \tag{3-218}$$

in Region (1) defined by the inequalities $H + \eta_1 < y < \infty$ and $-\infty < x < \infty$; in Region (2), $\eta_2 < y < H + \eta_1$ $-\infty < x < \infty$; and in Region (3), $-\infty < y < \eta_2$ $-\infty < x < \infty$. That is, we are dealing with a fluid sheet imbedded between two other fluids both with semi-infinite extent.

In the linear approximation the kinematic boundary condition on the upper interface between Regions (1) and (2) is

$$\frac{\partial}{\partial t}\, \eta_1(x,t) = -\frac{\partial}{\partial y}\, \phi_1(x,H,t) \quad , \tag{3-219}$$

and that on the lower interface between Regions (2) and (3) is

$$\frac{\partial}{\partial t}\, \eta_2(x,t) = -\frac{\partial}{\partial y}\, \phi_3(x,o,t) \quad . \tag{3-220}$$

The continuity of the y-component of the velocity vector on the upper interface leads to

$$\frac{\partial}{\partial y}\, \phi_1(x,H,t) = \frac{\partial}{\partial y}\, \phi_2(x,H,t) \quad , \tag{3-221}$$

whereas the same condition on the lower interface is

$$\frac{\partial}{\partial y}\, \phi_2(x,o,t) = \frac{\partial}{\partial y}\, \phi_3(x,o,t) \quad . \tag{3-222}$$

Pressure continuity requires that

$$\rho_2\, \frac{\partial}{\partial t}\, \phi_2(x,H,t) = \rho_1\, \frac{\partial}{\partial t}\, \phi_1(x,H,t) + (\rho_2 - \rho_1)\; g^*(t)\; \eta_1(x,t) - T_{s1}\, \frac{\partial^2}{\partial x^2}\, \eta_1(x,t) \tag{3-223}$$

on the upper interface, and

$$\rho_3\, \frac{\partial}{\partial t}\, \phi_3(x,o,t) = \rho_2\, \frac{\partial}{\partial t}\, \phi_2(x,o,t) + (\rho_3 - \rho_2)\; g^*(t)\; \eta_2(x,t) - T_{s2}\, \frac{\partial^2}{\partial x^2}\, \eta_2(x,t) \tag{3-224}$$

on the lower interface, where ρ_i is the density of the ith region. We will assume that the system is at rest initially and that the arbitrary initial perturbations of the two interfaces are specified.

The resolution of this initial value, boundary value problem can be accomplished by deriving a set of two coupled ordinary differential equations for the Fourier transforms of the two interface displacements, which is then solved for specified time-dependent accelerations. The Fourier transforms of the three velocity potentials satisfy

$$\frac{\partial^2}{\partial y^2}\phi_i(k,y,t) - k^2\,\phi_i(k,y,t) = 0 \;, \; (i = 1,\,2,\,3) \quad . \tag{3-225}$$

To ensure bounded solutions at infinity we take solutions of these equations as

$$\phi_1(k,y,t) = A_1(k,t)\,e^{-ky} \quad , \tag{3-226}$$

$$\phi_2(k,y,t) = A_2(k,t)\,e^{-ky} + A_3(k,t)\,e^{ky} \quad , \tag{3-227}$$

and

$$\phi_3(k,y,t) = A_4(k,t)\,e^{ky} \quad . \tag{3-228}$$

The conditions that the y-component of the velocity vector be continuous on the two interfaces lead to

$$A_1(k,t)\,e^{-kH} = A_2(k,t)\,e^{-kH} - A_3(k,t)\,e^{kH} \tag{3-229}$$

and

$$A_4(k,t) = -\,A_2(k,t) + A_3(k,t) \quad . \tag{3-230}$$

When differentiated with respect to time the kinematic conditions on the two interfaces reduce to

$$\frac{\partial^2}{\partial t^2}\eta_1(k,t) = k\,\frac{\partial}{\partial t}A_1(k,t)\,e^{-kH} \tag{3-231}$$

and

$$\frac{\partial^2}{\partial t^2}\eta_2(k,t) = -\,k\,\frac{\partial}{\partial t}A_4(k,t) \quad . \tag{3-232}$$

Also, the interface pressure continuity conditions become

$$\rho_1\frac{\partial}{\partial t}A_1(k,t)\,e^{-kH} = \rho_2\left[\frac{\partial}{\partial t}A_2(k,t)\,e^{-kH} + \frac{\partial}{\partial t}A_3(k,t)\,e^{kH}\right] - \left[(\rho_2 - \rho_1)\,g^*(t) + T_{s1}\,k^2\right]\eta_1(k,t) \tag{3-233}$$

and

$$\rho_3\frac{\partial}{\partial t}A_4(k,t) = \rho_2\left[\frac{\partial}{\partial t}A_2(k,t) + \frac{\partial}{\partial t}A_3(k,t)\right] + \left[(\rho_3 - \rho_2)\,g^*(t) + T_{s2}\,k^2\right]\eta_2(k,t)\,\cdot \tag{3-234}$$

Solving Eqs. (3-229) and (3-230) for A_2 and A_3 and entering the results into Eqs. (3-233) and (3-234) produces

$$\rho_1\frac{\partial}{\partial t}A_1(k,t)\,e^{-kH} = \frac{\rho_2}{\Delta_o}\left[\left(1 + e^{-2kH}\right)\frac{\partial}{\partial t}A_1(k,t) + 2\,\frac{\partial}{\partial t}A_4(k,t)\right] - \left[(\rho_2 - \rho_1)\,g^*(t) + T_{s1}\,k^2\right]\eta_1(k,t) \quad , \tag{3-235}$$

where $\Delta_o = -\,2\sinh(kH)$ and

$$\rho_3\frac{\partial}{\partial t}A_4(k,t) = \frac{\rho_2}{\Delta_o}\left[2\,e^{-kH}\frac{\partial}{\partial t}A_1(k,t) + (e^{kH} + e^{-kH})\,\frac{\partial}{\partial t}A_4(k,t)\right] + \left[(\rho_3 - \rho_2)\,g^*(t) + T_{s2}\,k^2\right]\eta_2(k,t) \quad . \tag{3-236}$$

Resolving these last two relations for A_{1t} and A_{4t} and combining the results with Eqs. (3-231) and (3-232) gives a set of two ordinary differential equations for the Fourier transforms of the displacements of the two interfaces. This set reads as follows:

$$\frac{\partial^2}{\partial t^2}\eta_1(k,t) + \frac{1}{D\Delta_o}\left[\rho_3 - \rho_2 - (\rho_2 + \rho_3)\,e^{-2kH}\right]\left[k(\rho_2 - \rho_1)\,g^*(t) + T_s k^3\right]\eta_1(k,t) - \frac{2\rho_2}{D\Delta_o}\left[k(\rho_3 - \rho_2)\,g^*(t) + T_{s2}\,k^3\right]e^{-kH}\,\eta_2(k,t) = 0 \tag{3-237}$$

and

$$-\frac{2\rho_2\, e^{-kH}}{D\Delta_o}\left[(\rho_2 - \rho_1)\, kg^*(t) + T_{s1}\, k^3\right]\eta_1(k,t)$$

$$+ \frac{\partial^2}{\partial t^2}\eta_2(k,t) - \frac{1}{D\Delta_o}\left[\rho_1 + \rho_2\right.$$

$$\left.- (\rho_1 - \rho_2)\, e^{-2kH}\right]\left[(\rho_3 - \rho_2)\, kg^*(t)\right.$$

$$\left.+ T_{s2}\, k^3\right]\eta_2(k,t) = 0 \quad . \tag{3-238}$$

In these last two differential equations we use the definition

$$D = \frac{1}{\Delta_o^2}\left[\rho_1\rho_3\, e^{-kH}\,\Delta_o^2 + \rho_2\, e^{kH}\left(1 + e^{-2kH}\right)^2\right.$$

$$\left.- 4\rho_2^2\, e^{-kH} - (\rho_1 + \rho_3)\,\rho_2\,\Delta_o\left(1 + e^{-2kH}\right)\right] \quad . \tag{3-239}$$

Solutions of Eqs. (3-237) and (3-238) can be obtained by a method similar to that used for Eqs. (3-192) and (3-193). The procedure now discussed will be more general than that used previously in that it is applicable to all forms of time-dependent accelerations.

Obtaining solutions of Eqs. (3-237) and (3-238) for arbitrarily specified time-dependent accelerations can be reduced to finding the solutions for simpler differential equations for the basis functions in terms of which the solutions of these equations can be expressed. Let the functions F_1 and F_2 be two linearly independent solutions of

$$\frac{d^2F}{dt^2} + \mu^2\, kg^*(t)\, F = 0 \quad , \tag{3-240}$$

and let G_1 and G_2 be two linearly independent solutions of

$$\frac{d^2G}{dt^2} - \nu^2\, kg^*(t)\, G = 0 \quad . \tag{3-241}$$

Let c_i, $i = 1, 2, 3$, and 4 be four arbitrary constants; then the general solution of Eqs. (3-237) and (3-238) in vanishing surface tensions can be written in the form

$$\eta_1(k,t) = c_1\, F_1 + c_2\, F_2 + c_3\, G_1 + c_4\, G_2 \tag{3-242}$$

and

$$\eta_2(k,t) = S_1\left[c_1\, F_1 + c_2\, F_2\right] + S_2\left[c_3\, G_1 + c_4\, G_2\right] , \tag{3-243}$$

provided that the following conditions are satisfied. The constant μ^2 is a root of the quadratic

$$\mu^4 + \frac{\mu^2}{D\Delta}\left\{(\rho_3 - \rho_2)\left[\rho_1 + \rho_2 - (\rho_1 - \rho_2)\, e^{-2kH}\right]\right.$$

$$\left.- (\rho_2 - \rho_1)\left[\rho_3 - \rho_2 - (\rho_2 + \rho_3)\, e^{-2kH}\right]\right\}$$

$$- \frac{(\rho_2 - \rho_1)\,(\rho_3 - \rho_2)}{(D\Delta)^2}\left\{4\rho_2^2\, e^{-2kH}\right.$$

$$+ \left[\rho_3 - \rho_2 - (\rho_2 + \rho_3)\, e^{-2kH}\right]\left[\rho_1 + \rho_2\right.$$

$$\left.\left.- (\rho_1 - \rho_2)\, e^{-2kH}\right]\right\} = 0 \quad . \tag{3-244}$$

The constant ν^2 is a root of the quadratic obtained from Eq. (3-244) by replacing μ^2 with $-\nu^2$. The constant S_1 is given by either

$$S_1 = \frac{-\mu^2 D\Delta + (\rho_2 - \rho_1)\left[\rho_3 - \rho_2 - (\rho_2 + \rho_3)\, e^{-2kH}\right]}{2\rho_2\,(\rho_3 - \rho_2)\, e^{-kH}} \tag{3-245}$$

or

$$S_1 = \frac{-2\rho_2\left(\rho_2 - \rho_1\right) e^{-kH}}{\mu^2 D\Delta + \left(\rho_3 - \rho_2\right)\left[\rho_1 + \rho_2 - \left(\rho_1 - \rho_2\right) e^{-2kH}\right]} \ . \tag{3-246}$$

The constant S_2 is given by either

$$S_2 = \frac{\nu^2 D\Delta + \left(\rho_2 - \rho_1\right)\left[\rho_3 - \rho_2 - \left(\rho_2 + \rho_3\right) e^{-2kH}\right]}{2\rho_2\left(\rho_3 - \rho_2\right) e^{-kH}} \tag{3-247}$$

or

$$S_2 = \frac{-2\rho_2\left(\rho_2 - \rho_1\right) e^{-kH}}{-\nu^2 D\Delta + \left(\rho_3 - \rho_2\right)\left[\rho_1 + \rho_2 - \left(\rho_1 - \rho_2\right) e^{-2kH}\right]} \ . \tag{3-248}$$

The four arbitrary constants that appear in the general solutions for the Fourier transforms of the interface displacements contained in Eqs. (3-242) and (3-243) can be determined in terms of the initial conditions. For example, for a system at rest initially, the arbitrary constants satisfy

$$c_1 F_{1o} + c_2 F_{2o} + c_3 G_{1o} + c_4 G_{2o} = \eta_1(k,o) , \tag{3-249}$$

$$S_1\left[c_1 F_{1o} + c_2 F_{2o}\right] + S_2\left[c_3 G_{1o} + c_4 G_{2o}\right] = \eta_2(k,o) \ , \tag{3-250}$$

$$c_1 F_{1to} + c_2 F_{2to} + c_3 G_{1to} + c_4 G_{2to} = 0 \ , \tag{3-251}$$

and

$$S_1\left[c_1 F_{1to} + c_2 F_{2to}\right] + S_2\left[c_3 G_{1to} + c_4 G_{2to}\right] = 0 \ . \tag{3-252}$$

In these last four relations the subscript o indicates that the basis functions and their derivatives with respect to time are to be evaluated at $t = 0$. From Eqs. (3-251) and (3-252) it follows that

$$c_2 = -c_1 \frac{F_{1to}}{F_{2to}} \tag{3-253}$$

and that

$$c_4 = -c_3 \frac{G_{1to}}{G_{2to}} \ . \tag{3-254}$$

Entering Eqs. (3-253) and (3-254) into Eqs. (3-249) and (3-250) and solving the results for the constants c_1 and c_3 produces

$$c_1 = \frac{F_{2to}\left[S_2\,\eta_1(k,o) - \eta_2(k,o)\right]}{\left(S_2 - S_1\right)\left(F_{2to} F_{1o} - F_{2o} F_{1to}\right)} \tag{3-255}$$

and

$$c_3 = \frac{G_{2to}\left[-S_1\,\eta_1(k,o) + \eta_2(k,o)\right]}{\left(S_2 - S_1\right)\left(G_{2to} G_{1o} - G_{2o} G_{1to}\right)} \ . \tag{3-256}$$

In view of Eqs. (3-253)-(3-256) the solutions contained in Eqs. (3-242) and (3-243) for the Fourier transforms of the interface displacements become

$$\eta_1(k,t) = \frac{\left(F_{2to} F_1 - F_{1to} F_2\right)}{\left(F_{2to} F_{1o} - F_{1to} F_{2o}\right)} \cdots$$
$$\cdots \frac{\left[S_2\,\eta_1(k,o) - \eta_2(k,o)\right]}{\left(S_2 - S_1\right)}$$
$$+ \frac{\left(G_{2to} G_1 - G_{1to} G_2\right)}{\left(G_{2to} G_{1o} - G_{1to} G_{2o}\right)} \cdots$$
$$\cdots \frac{\left[-S_1\,\eta_1(k,o) + \eta_2(k,o)\right]}{\left(S_2 - S_1\right)} \tag{3-257}$$

and

$$\eta_2(k,t) = \frac{\left(F_{2to}\, F_1 - F_{1to}\, F_2\right)}{\left(F_{2to}\, F_{1o} - F_{1to}\, F_{2o}\right)} \cdots$$

$$\cdots \frac{S_1 \left[S_2\, \eta_1(k,o) - \eta_2(k,o) \right]}{\left(S_2 - S_1\right)}$$

$$+ \frac{\left(G_{2to}\, G_1 - G_{1to}\, G_2\right)}{\left(G_{2to}\, G_{1o} - G_{1to}\, G_{2o}\right)} \cdots$$

$$\cdots \frac{S_2 \left[- S_1\, \eta_1(k,o) + \eta_2(k,o) \right]}{\left(S_2 - S_1\right)} \quad . \qquad (3\text{-}258)$$

The results obtained in these last two relations are valid regardless of the actual time variations of the acceleration. For each particular functional dependence of the acceleration upon time, the basis functions required for the solutions found in Eqs. (3-257) and (3-258) are determined as solutions of Eqs. (3-240) and (3-241). Explicit results for the basis functions are included in Table 1 for both constant accelerations and for time-dependent accelerations of the form given in Eq. (3-101).

4. SOLUTIONS OF TAYLOR INSTABILITY INITIAL VALUE PROBLEMS FOR VISCOUS FLUIDS

Solutions of the Taylor instability initial value problems are worked out for half-spaces of a viscous fluid, sheets of viscous fluids, and double half-spaces of viscous fluids. Both the stream function and the DNS formulations discussed in Sec. 2 are used to solve these problems.

4.1. Solutions for a Half-Space Configuration with Constant Acceleration

4.1.1. Application of the Stream Function Formulation

As shown in Sec. 2.1 we want to solve

$$\rho \frac{\partial}{\partial t} \nabla^2 \psi = \mu \nabla^2 \nabla^2 \psi \quad , \qquad (4\text{-}1)$$

subject to appropriate boundary, kinematic, and initial conditions. In the linear approximation the kinematic condition on the surface of a half-space taken as the lower half-plane becomes

$$\frac{\partial}{\partial t} \eta(x,t) = - \frac{\partial^2}{\partial x \partial y} \psi(x,o,t) \quad . \qquad (4\text{-}2)$$

Boundary conditions to be satisfied on the interface arise from the components of the stress tensor. For a fluid that occupies the lower half-plane the τ_{yy}-component of the stress tensor satisfies

$$\tau_{yy} = \frac{T_s \frac{\partial^2}{\partial x^2} \eta(x,t)}{\left\{ 1 + \left[\frac{\partial}{\partial x} \eta(x,t) \right]^2 \right\}^{3/2}} \qquad (4\text{-}3)$$

on the surface, and the τ_{yx}-component satisfies

$$\tau_{yx} = 0 \quad . \qquad (4\text{-}4)$$

TABLE I

BASIS FUNCTIONS FOR VARIOUS ACCELERATIONS

Form of the Acceleration	F_1	F_2	G_1	G_2
$g^*(t) = g_o$	$\cos(\mu t \sqrt{kg_o})$	$\sin(\mu t \sqrt{kg_o})$	$\cosh(\nu t \sqrt{kg_o})$	$\sinh(\nu t \sqrt{kg_o})$
$g^*(t) = g_o \left(1 - \frac{t}{T}\right)^{2r-2}$	$\tau^{1/2} J_{\frac{1}{2r}} \left(\frac{\mu T}{r} \sqrt{kg_o\, \tau^r} \right)$ with $\tau = 1 - \frac{t}{T}$	$\tau^{1/2} J_{-\frac{1}{2r}} \left(\frac{\mu T}{r} \sqrt{kg_o\, \tau^r} \right)$ with $\tau = 1 - \frac{t}{T}$	$\tau^{1/2} I_{\frac{1}{2r}} \left(\frac{\nu T}{r} \sqrt{kg_o\, \tau^r} \right)$ with $\tau = 1 - \frac{t}{T}$	$\tau^{1/2} I_{-\frac{1}{2r}} \left(\nu \frac{T}{r} \sqrt{kg_o\, \tau^r} \right)$ with $\tau = 1 - \frac{t}{T}$

Entering the expressions for the components of the stress tensor into Eqs. (4-3) and (4-4) yields

$$- p + 2\mu \frac{\partial v}{\partial y} + \lambda \operatorname{div} \vec{V} = \frac{T_s \frac{\partial^2 \eta}{\partial x^2}(x,t)}{\left\{1 + \left[\frac{\partial}{\partial x} \eta(x,t)\right]^2\right\}^{3/2}} , \tag{4-5}$$

where μ and λ are the viscosity coefficients, and

$$\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = 0 . \tag{4-6}$$

Upon setting the arbitrary function of time equal to zero in the pressure field expressions of Eq. (2-19) and assuming an incompressible fluid so that $\operatorname{div} \vec{V} = 0$ in Eq. (4-5), we find upon combining these two equations in the linear approximation that

$$\rho \frac{\partial^2}{\partial t \partial x} \psi(x,o,t) - \rho g^* \eta(x,t) + \mu \left[\frac{\partial^3}{\partial x \partial y^2} \psi(x,o,t) - \frac{\partial^3}{\partial x^3} \psi(x,o,t)\right] = 0 \tag{4-7}$$

is the linear boundary condition that comes out of the τ_{yy}-component of the stress tensor for zero surface tension. Also, Eq. (4-6) becomes

$$\frac{\partial^3 \psi}{\partial y^3}(x,o,t) = \frac{\partial^3}{\partial x^2 \partial y} \psi(x,o,t) \tag{4-8}$$

in the linear approximation.

To solve the boundary value problem defined by Eqs. (4-1), (4-2), (4-7), and (4-8) we introduce the Laplace-Fourier transform

$$\psi(k,y,s) = \int_0^\infty dt\, e^{-st} \int_{-\infty}^\infty dx\, e^{ikx}\, \psi(x,y,t) . \tag{4-9}$$

Let $\nu = \mu/\rho$ be the kinematic viscosity; then the Laplace-Fourier transform of Eqs. (4-1), (4-2), (4-7), and (4-8) for a system at rest initially produces the following relations:

$$\frac{d^4}{dy^4} \psi(k,y,s) - \left(\frac{s}{\nu} + 2k^2\right) \frac{d^2}{dy^2} \psi(k,y,s) + \left(k^4 + \frac{sk^2}{\nu}\right) \psi(k,y,s) = 0 , \tag{4-10}$$

$$s\, \eta(k,s) - \eta(k,o) = ik \frac{d}{dy} \psi(k,o,s) , \tag{4-11}$$

$$(iks/\nu)\, \psi(k,o,s) + (g^*/\nu)\, \eta(k,s) + ik \frac{d^2}{dy^2} \psi(k,o,s) + ik^3 \psi(k,o,s) = 0 , \tag{4-12}$$

and

$$\frac{d^3}{dy^3} \psi(k,o,x) = - k^2 \frac{d}{dy} \psi(k,o,s) . \tag{4-13}$$

The fourth-order ordinary differential equation that appears in Eq. (4-10) is of the so-called Orr-Sommerfeld type. Its general solution can be written as

$$\psi(k,y,s) = A_1(k,s) \exp(ky) + A_2(k,s) \exp\left(y\sqrt{k^2 + \frac{s}{\nu}}\right) A_3(k,s) \exp(-ky) + A_4(k,s) \exp\left(-y\sqrt{k^2 + \frac{s}{\nu}}\right) . \tag{4-14}$$

For a half-space in the lower half-plane we take $A_3 = 0 = A_4$ for bounded solutions at infinity and write

$$\psi(k,y,s) = A_1(k,s)\, e^{ky} + A_2(k,s)\, e^{Ky} \tag{4-15}$$

with

$$K = \sqrt{k^2 + \frac{s}{\nu}} . \tag{4-16}$$

At this point three quantities, namely, $A_1(k,s)$ and $A_2(k,s)$ in Eq. (4-15) and $\eta(k,s)$ remain to be found. Entering Eq. (4-15) into Eqs. (4-11)-(4-13) gives three algebraic equations for their determination. It follows from Eq. (14-13) that

$$A_2(k,s) = \frac{-2k^3}{k^2 K + K^3} A_1(k,s) \quad , \tag{4-17}$$

and Eqs. (4-11) and (4-12) become

$$s\,\eta(k,s) - \eta(k,o) = (ik) \left[k A_1(k,s) + K A_2(k,s)\right] \tag{4-18}$$

and

$$(g^*/\nu)\ \eta(k,s) = (-ik) \left\{ \left(\frac{s}{\nu} + k^2\right) \left[A_1(k,s) + A_2(k,s)\right] + k^2 A_1(k,s) + K^2 A_2(k,s) \right\} . \tag{4-19}$$

Upon resolving Eqs. (4-17)-(4-19) the Laplace-Fourier transform of the interface displacement is found to be given by

$$\frac{\eta(k,s)}{\eta(k,o)} = \frac{1}{s} \cdot \frac{k^4 \left[\left(2 + \frac{s}{\nu k^2}\right)^2 - 4\sqrt{1 + \frac{s}{\nu k^2}} \right]}{k^4 \left[\left(2 + \frac{s}{\nu k^2}\right)^2 - 4\sqrt{1 + \frac{s}{\nu k^2}} \right] + \frac{kg^*}{\nu^2}} . \tag{4-20}$$

Also, the Laplace-Fourier transform of the partial derivative of the interface displacement with respect to time is

$$s\,\eta(k,s) - \eta(k,o) = \frac{-\frac{k}{\nu^2} g^* \eta(k,o)}{k^4 \left[\left(2 + \frac{s}{\nu k^2}\right)^2 - 4\sqrt{1 + \frac{s}{\nu k^2}} \right] + \frac{kg^*}{\nu^2}} . \tag{4-21}$$

The space-time response of the interface displacement and its time derivative can be determined from Eqs. (4-20) and (4-21) by using the inversion theorems for Laplace and Fourier transforms. However, before considering this aspect of the problem we shall discuss the application of the DNS formulation presented in Sec. 2.3 to this same problem.

4.1.2. Application of the DNS Formulation

As indicated in Sec. 2.3 the DNS formulation provides a unified approach for the treatment of Taylor instability initial value problems for both viscous and inviscid fluids. In contrast to the stream function method of Sec. 4.1.1, where it was necessary to solve a fourth-order differential equation, only second-order differential equations need to be solved in the DNS formulation. We now demonstrate this for the half-space problem.

We start from the complex Fourier transforms of Eqs. (2-57) and (2-59), which are

$$\frac{\partial^2}{\partial y^2} P(k,y,t) - k^2 P(k,y,t) = 0 \quad , \tag{4-22}$$

$$\frac{\partial}{\partial t} u(k,y,t) = \frac{ik}{\rho} P(k,y,t) + \nu \left[\frac{\partial^2}{\partial y^2} u(k,y,t) - k^2 u(k,y,t) \right] , \tag{4-23}$$

and

$$\frac{\partial}{\partial t} v(k,y,t) = -\frac{1}{\rho} \frac{\partial}{\partial y} P(k,y,t) + \nu \left[\frac{\partial^2}{\partial y^2} v(k,y,t) - k^2 v(k,y,t) \right] . \tag{4-24}$$

The solution of Eq. (4-22) of interest for the lower half-space problem is

$$P(k,y,t) = A_1(k,t)\ e^{ky} \quad , \tag{4-25}$$

with which Eqs. (4-23) and (4-24) become

$$\frac{\partial}{\partial t} u(k,y,t) = \frac{ik}{\rho} A_1(k,t)\ e^{ky} + \nu \left[\frac{\partial^2}{\partial y^2} u(k,y,t) - k^2 u(k,y,t) \right] \tag{4-26}$$

and

$$\frac{\partial}{\partial t} v(k,y,t) = -\frac{k}{\rho} A_1(k,t)\ e^{ky} + \nu \left[\frac{\partial^2}{\partial y^2} v(k,y,t) - k^2 v(k,y,t) \right] . \tag{4-27}$$

In this formulation the kinematic condition on the interface is used in the form

$$\frac{\partial}{\partial t} \eta(x,t) = v(x,o,t) \quad , \tag{4-28}$$

the boundary condition from the τ_{yy}-component of the stress tensor is

$$P(x,o,t) - \rho g^* \eta(x,t) - 2\mu \frac{\partial}{\partial y} v(x,o,t) = 0 \quad , \tag{4-29}$$

and the boundary condition from the τ_{yx}-component of the stress tensor is

$$\frac{\partial}{\partial y} u(x,o,t) + \frac{\partial}{\partial x} v(x,o,t) = 0 \quad . \tag{4-30}$$

The Laplace transforms of the complex Fourier transforms of the x- and y- components of the linearized Navier-Stokes equations obtained from Eqs. (4-26) and (4-27) are

$$s\, u(k,y,s) - u(k,y,o) = \frac{ik}{\rho} A_1(k,s)\, e^{ky} + \nu\left[\frac{\partial^2}{\partial y^2} u(k,y,s) - k^2 u(k,y,s)\right] \tag{4-31}$$

and

$$s\, v(k,y,s) - v(k,y,o) = -\frac{k}{\rho} A_1(k,s)\, e^{ky} + \nu\left[\frac{\partial^2}{\partial y^2} v(k,y,s) - k^2 v(k,y,s)\right] \quad . \tag{4-32}$$

If the fluid is at rest initially, then Eqs. (4-31) and (4-32) reduce to the following one-dimensional, inhomogeneous scalar Helmholtz equations for the Laplace-Fourier transforms of the x- and y-components of the velocity vector;

$$\frac{\partial^2}{\partial y^2} u(k,y,s) - \left(\frac{s}{\nu} + k^2\right) u(k,y,s) = -\frac{ik}{\rho\nu} A_1(k,s)\, e^{ky} \tag{4-33}$$

and

$$\frac{\partial^2}{\partial y^2} v(k,y,s) - \left(\frac{s}{\nu} + k^2\right) v(k,y,s) = \frac{k}{\rho\nu} A_1(k,s)\, e^{ky} \quad . \tag{4-34}$$

With the quantity K, as defined in Eq. (4-16), the solutions of Eqs. (4-33) and (4-34) appropriate for a half-space in the lower half-plane are

$$u(k,y,s) = C_1(k,s)\, e^{Ky} + \frac{ik}{\rho s} A_1(k,s)\, e^{ky} \tag{4-35}$$

and

$$v(k,y,s) = C_3(k,s)\, e^{Ky} - \frac{k}{\rho s} A_1(k,s)\, e^{ky} \quad . \tag{4-36}$$

The quantities C_1, C_3, and A_1 in these last two equations are found by satisfying the Laplace-Fourier transforms of the kinematic and boundary conditions, together with that of the continuity equation, which is

$$\frac{\partial}{\partial y} v(k,y,s) = i\, k\, u(k,y,s) \quad . \tag{4-37}$$

Upon entering Eqs. (4-35) and (4-36), this relation leads to

$$C_3(k,s) = (ik/K)\, C_1(k,s) \quad . \tag{4-38}$$

The Laplace-Fourier transforms of the kinematic and boundary conditions read

$$A_1(k,s) = P(k,o,s) = \rho g^* \eta(k,s) + 2\mu \frac{\partial}{\partial y} v(k,o,s) \quad , \tag{4-39}$$

$$\frac{\partial}{\partial y} u(k,o,s) = ik\, v(k,o,s) \quad , \tag{4-40}$$

and

$$s\, \eta(k,s) - \eta(k,o) = v(k,o,s) \quad . \tag{4-41}$$

These last three equations become

$$A_1(k,s) = \rho g^* \eta(k,s) + 2\mu\left[K\, C_3(k,s) - \frac{k^2}{\rho s} A_1(k,s)\right] , \tag{4-42}$$

$$K\, C_1(k,s) + \frac{ik^2}{\rho s} A_1(k,s) = (ik)\left[C_3(k,s) - \frac{k}{\rho s} A_1(k,s)\right] , \tag{4-43}$$

and

$$s\, \eta(k,s) = \eta(k,o) + C_3(k,s) - \frac{k}{\rho s} A_1(k,s) \quad . \tag{4-44}$$

The same result for the Laplace-Fourier transform of the interface displacement as that quoted in Eq. (4-20) is obtained by solving Eqs. (4-38) and (4-42)-(4-44).

4.1.3. Inversion of the Laplace Transform

In Secs. 4.1.1 and 4.1.2 the stream function and the DNS formulations were used to determine the Laplace-Fourier transform of the displacement of the surface of a half-space of a viscous fluid subjected to a time-independent acceleration. Also, the Laplace-Fourier transform of the time derivative of the surface perturbation is given by Eq. (4-21). If this equation is inverted back into the time domain, the time derivative of the Fourier transform of the surface displacement is found to be

$$\frac{\partial}{\partial t}\, \eta(k,t) = \eta(k,o)\left(-\frac{g^*}{\nu k}\right) \sum_{i=1}^{4} \frac{q_i}{D'\left(q_i\right)}\, e^{\nu k^2 t\left(q_i^2 - 1\right)}\, \mathrm{erfc}\left(-q_i\sqrt{\nu k^2 t}\right) \quad . \tag{4-45}$$

In this result the quantities q_i, i = 1, 2, 3, and 4 are the four roots of

$$q_i^4 + 2q_i^2 - 4q_i + 1 + \frac{g^*}{\nu^2 k^3} = 0 \quad , \tag{4-46}$$

and the denominator is the cubic

$$D'(q_i) = \left(4\, q_i^3 + q_i - 1\right) \quad . \tag{4-47}$$

To establish Eq. (4-45) from Eq. (4-21) the following procedure can be used. First write the right-hand side of Eq. (4-21) as

$$\frac{\eta(k,o)\ (-k/\nu^2) g^*}{\dfrac{\left(s + \nu k^2 + \nu k^2\right)^2}{\nu^2} - \dfrac{4k^3}{\sqrt{\nu}}\sqrt{s + \nu k^2} + \dfrac{kg^*}{\nu^2}} \quad .$$

To find the inverse of this expression multiply the inverse of the following expressions by $\exp(-t\nu k^2)$;

$$\frac{\eta(k,o)\ (-k/\nu^2) g^*}{\dfrac{\left(s + \nu k^2\right)^2}{\nu^2} - \dfrac{4k^3}{\sqrt{\nu}}\sqrt{s} + \dfrac{kg^3}{\nu^2}} = \eta(k,o)\left(-\frac{k}{\nu^2}\right)\frac{g^*(\nu k^2)}{k^4}\, \frac{1}{\left(\nu k^2\right)\left[\left(\dfrac{s}{\nu k^2} + 1\right)^2 - 4\sqrt{\dfrac{s}{\nu k^2}} + \dfrac{g^*}{\nu^2 k^3}\right]} \quad . \tag{4-48}$$

We now invert the following quantity and replace t with $\nu k^2 t$ in the result

$$\eta(k,0)\left(-\frac{k}{\nu^2}\right)\frac{g^*(\nu k^2)}{k^4}\, \frac{1}{(s+1)^2 - 4\sqrt{s} + \dfrac{g^*}{\nu^2 k^3}} \quad .$$

Let

$$D(q) = q^4 + 2q^2 - 4q + 1 + \frac{g^*}{\nu^2 k^3} \quad , \qquad (4\text{-}49)$$

and let q_i be a root of $D(q_i) = 0$, as indicated in Eq. (4-46). Then we can write

$$\frac{1}{D(q)} = \sum_{i=1}^{4} \frac{1}{D'\left(q_i\right)\left(q - q_i\right)} = \sum_{i=1}^{4} \frac{1}{D'\left(q_i\right)\left(\sqrt{s} - q_i\right)}, \qquad (4\text{-}50)$$

where the derivative is given by Eq. (4-47). The inverse Laplace transform of the reciprocal of the factor $\sqrt{s} - q_i$ is

$$\frac{1}{\sqrt{\pi t}} + q_i \exp\left(q_i^2 t\right) \operatorname{erfc}\left(- q_i\sqrt{t}\right)$$

in terms of the complimentary error function. Consequently, the Laplace transform inverse of Eq. (4-50) is

$$\sum_{i=1}^{4} \frac{1}{D'\left(q_i\right)\left(\sqrt{s} - q_i\right)} \xRightarrow{\text{INVERSION}} \sum_{i=1}^{4} \frac{q_i}{D'\left(q_i\right)} e^{tq_i^2} \operatorname{erfc}(q_i\sqrt{t}) \quad , \qquad (4\text{-}58)$$

$$\frac{1}{\sqrt{\pi t}} \sum_{i=1}^{4} \frac{1}{D'\left(q_i\right)} + \sum_{i=1}^{4} \frac{q_i}{D'\left(q_i\right)} \exp\left(q_i^2 t\right) \operatorname{erfc}\left(-q_i\sqrt{t}\right) \quad .$$

Because we can also write

$$\frac{1}{\prod_{i=1}^{4}\left(q - q_i\right)} = \sum_{i=1}^{4} \frac{\alpha_1}{\left(q - q_i\right)} \quad , \qquad (4\text{-}51)$$

where

$$\alpha_1 = \frac{1}{D'\left(q_i\right)} \quad , \qquad (4\text{-}52)$$

or

$$\alpha_1 = \frac{1}{\left(q_1 - q_2\right)\left(q_1 - q_3\right)\left(q_1 - q_4\right)} \quad , \qquad (4\text{-}53)$$

$$\alpha_2 = \frac{1}{\left(q_2 - q_1\right)\left(q_2 - q_3\right)\left(q_2 - q_4\right)} \quad , \qquad (4\text{-}54)$$

$$\alpha_3 = \frac{1}{\left(q_3 - q_1\right)\left(q_3 - q_2\right)\left(q_3 - q_4\right)} \quad , \qquad (4\text{-}55)$$

and

$$\alpha_4 = \frac{1}{\left(q_4 - q_1\right)\left(q_4 - q_2\right)\left(q_4 - q_3\right)} \quad , \qquad (4\text{-}56)$$

it follows that

$$\sum_{i=1}^{4} \frac{1}{D'\left(q_i\right)} = 0 \quad . \qquad (4\text{-}57)$$

Thus, we have

and the inverse of the right-hand side of Eq. (4-48) is

$$\eta(k,o)\left(- \frac{g^*}{\nu k}\right)\sum_{i=1}^{4} \frac{q_i}{D'\left(q_i\right)} e^{\nu k^2 t q_i^2} \operatorname{erfc}\left(- q_i\sqrt{\nu k^2 t}\right) .$$

Multiplying this expression by the exponential $\exp(- t\nu k^2)$ gives the result quoted in Eq. (4-45) for the Fourier transform of the time derivative of the displacement of the surface.

4.2. Solution for a Viscous Fluid Sheet with Constant Acceleration

In the linear approximation of the stream function formulation we have to solve

$$\rho \frac{\partial}{\partial t} \nabla^2\psi = \mu\nabla^2\nabla^2\psi \qquad (4\text{-}59)$$

to determine the space-time response of the upper and lower surfaces of a viscous fluid sheet. The pressure field can be written as

$$p(x,y,t) = p_o + \rho \frac{\partial^2}{\partial t \partial x} \psi(x,y,t) - \rho g^* y - \mu\left[\frac{\partial^3}{\partial x^3} \psi(x,y,t) + \frac{\partial^3}{\partial x \partial y^2} \psi(x,y,t)\right] \tag{4-60}$$

and

$$p(x,y,t) = p_H + \rho \frac{\partial^2}{\partial t \partial x} \psi(x,y,t) - \rho g^*(y - H) - \mu\left[\frac{\partial^3}{\partial x^3} \psi(x,y,t) + \frac{\partial^3}{\partial x \partial y^2} \psi(x,y,t)\right] . \tag{4-61}$$

In Eq. (4-60) the pressure at the mean position of the lower surface is p_o, and p_H is the pressure at the mean position of the upper surface in Eq. (4-61). With these two expressions for the pressure field it is possible to put the τ_{yy}-component of the stress tensor for an incompressible fluid, namely,

$$\tau_{yy} = - p - 2\mu \frac{\partial v}{\partial y} \tag{4-62}$$

into two alternative forms that can be used in the boundary conditions on the upper and lower surfaces of the fluid sheet.

The kinematic condition on the upper surface at $y = H + \eta_1(x,t)$ is

$$\frac{\partial}{\partial t} \eta_1(x,t) = v(x,H,t) = - \frac{\partial^2}{\partial x \partial y} \psi(x,H,t) \quad , \tag{4-63}$$

and that on the lower surface at $y = \eta_2(x,t)$ is

$$\frac{\partial}{\partial t} \eta_2(x,t) = v(x,o,t) = - \frac{\partial^2}{\partial x \partial y} \psi(x,o,t) \quad , \tag{4-64}$$

both in the linear approximation. The τ_{yy}-component of the stress tensor leads to the boundary condition

$$\frac{\partial^2}{\partial t \partial x} \psi(x,o,t) = g^* \eta_2(x,t) + \nu \left[\frac{\partial^3}{\partial x^3} \psi(x,o,t) - \frac{\partial^3}{\partial x \partial y^2} \psi(x,o,t)\right] \tag{4-65}$$

on the lower surface and to

$$\frac{\partial^2}{\partial t \partial x} \psi(x,H,t) = g^* \eta_1(x,t) + \nu\left[\frac{\partial^3}{\partial x^3} \psi(x,H,t) - \frac{\partial^3}{\partial x \partial y^2} \psi(x,H,t)\right] \tag{4-66}$$

on the upper surface. The τ_{yx}-component of the stress tensor becomes

$$\frac{\partial^3}{\partial y^3} \psi(x,o,t) = \frac{\partial^3}{\partial x^2 \partial y} \psi(x,o,t) \tag{4-67}$$

on the lower surface and

$$\frac{\partial^3}{\partial y^3} \psi(x,H,t) = \frac{\partial^3}{\partial x^2 \partial y} \psi(x,H,t) \tag{4-68}$$

on the supper surface.

The solution of Eq. (4-59), subject to the kinematic and boundary conditions of Eqs. (4-63)-(4-68), can be determined by the method of multiple integral transforms. We use a Laplace transform on the time coordinate and a complex Fourier transform on the x-coordinate, namely,

$$\psi(k,y,s) = \int_0^\infty dt \, e^{-st} \int_{-\infty}^\infty dk \, e^{ikx} \, \psi(x,y,t) \ . \tag{4-69}$$

The Laplace-Fourier transform of Eq. (4-59) is

$$\frac{d^4}{dy^4} \psi(k,y,s) - \left(\frac{s}{\nu} + 2k^2\right) \frac{d^2}{dy^2} \psi(k,y,s) + k^2 \left(k^2 + \frac{s}{\nu}\right) \psi(k,y,s) = 0 \ . \tag{4-70}$$

The general solution of this equation is

$$\psi(k,y,s) = A_1(k,s) \, e^{ky} + A_2(k,s) \, e^{-ky} + A_3(k,s) \, e^{Ky} + A_4(k,s) \, e^{-Ky} \quad , \tag{4-71}$$

where $K^2 = k^2 + s/\nu$.

At this point the problem entails the calculation of six quantities. These are the A_i, i = 1, 2, 3, and 4 in Eq. (4-71) and the Laplace-Fourier transforms of the displacements of the upper and lower surfaces. By taking the Laplace-Fourier transforms of the kinematic and boundary conditions, i.e., of Eqs. (4-63)-(4-68), six relations from which the remaining six unknowns can be found are obtained. This reduction leads to an algebraic problem involving six inhomogeneous algebraic equations whose inhomogeneous terms contain the Fourier transforms of the arbitrary initial perturbations of the upper and lower surfaces of the fluid sheet.

The Laplace-Fourier transforms of Eqs. (4-63)-(4-68) give the following six relations:

$$s\eta_1(k,s) - \eta_1(k,o) = ik \frac{d}{dy} \psi(k,H,s) \quad , \tag{4-72}$$

$$s\eta_2(k,s) - \eta_2(k,o) = ik \frac{d}{dy} \psi(k,o,s) \quad , \tag{4-73}$$

$$(ik) \frac{d^2}{dy^2} \psi(k,o,s) + (ik)\, K^2\, \psi(k,o,s) + \frac{g^*}{\nu} \eta_2(k,s) = 0 \quad , \tag{4-74}$$

$$(ik) \frac{d^2}{dy^2} \psi(k,H,s) + (ik)\, K^2\, \psi(k,H,s) + \frac{g^*}{\nu} \eta_1(k,s) = 0 \quad , \tag{4-75}$$

$$\frac{d^3}{dy^3} \psi(k,o,s) = - k^2 \frac{d}{dy} \psi(k,o,s) \quad , \tag{4-76}$$

and

$$\frac{d^3}{dy^3} \psi(k,H,s) = - k^2 \frac{d}{dy} \psi(k,H,s) \quad . \tag{4-77}$$

If we define the quantities $X_j(k,s)$ by

$$X_j(k,s) = (ik)\, A_j(k,s) \quad , 1 \leq j \leq 4 \quad , \tag{4-78}$$

and enter Eq. (4-71) into Eqs. (4-72)-(4-77), we obtain the algebraic set

$$A \begin{pmatrix} \eta_1(k,s) \\ \eta_2(k,s) \\ X_1(k,s) \\ X_2(k,s) \\ X_3(k,s) \\ X_4(k,s) \end{pmatrix} = \begin{pmatrix} \eta_1(k,o) \\ \eta_2(k,o) \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \quad , \tag{4-79}$$

where the matrix A is given by

$$A = \begin{bmatrix} p & 0 & -ke^{-kH} & ke^{-kH} & -Ke^{KH} & Ke^{-KH} \\ 0 & p & -k & k & -K & K \\ g^*/\nu & 0 & (k^2+K^2)e^{kH} & (k^2+K^2)e^{-kH} & 2K^2e^{KH} & 2K^2e^{-KH} \\ 0 & g^*/\nu & k^2+K^2 & k^2+K^2 & 2K^2 & 2K^2 \\ 0 & 0 & 2k^3e^{kH} & -2k^3e^{-kH} & K(k^2+K^2)e^{KH} & -K(k^2+K^2)e^{-KH} \\ 0 & 0 & 2k^3 & -2k^3 & K(k^2+K^2) & -K(k^2+K^2) \end{bmatrix} \quad . \tag{4-80}$$

The solutions of Eq. (4-79) for the Laplace-Fourier transform of the displacement of the upper surface of the fluid sheet can be written in the form

$$\eta_1(k,s) = \frac{\eta_1(k,o)\left[s\,\delta_{11}(k,s) + \frac{g^*}{\nu} M_{31}(k,s)\right] - \eta_2(k,o)\,\frac{g^*}{\nu}\,T_{31}(k,s)}{\delta_{11}(k,s)s^2 + \left(\frac{g^*}{\nu}\right)^2 D_{31}(k,s) + s\,\frac{g^*}{\nu}\left[\delta_{31}(k,s) - D_{21}(k,s)\right]} \quad . \tag{4-81}$$

The determinants in this result are defined by the relations that follow:

$$\delta_{11}(k,s) = \begin{vmatrix} (k^2+K^2)e^{kH} & (k^2+K^2)e^{-kH} & 2K^2e^{KH} & 2K^2e^{-KH} \\ k^2+K^2 & k^2+K^2 & 2K^2 & 2K^2 \\ 2k^3e^{kH} & -2k^3e^{-kH} & K(k^2+K^2)e^{KH} & -K(k^2+K^2)e^{-KH} \\ 2k^3 & -2k^3 & K(k^2+K^2) & -K(k^2+K^2) \end{vmatrix} , \tag{4-82}$$

$$M_{31}(k,s) = \begin{vmatrix} -k & k & -K & K \\ (k^2+K^2)e^{kH} & (k^2+K^2)e^{-kH} & 2K^2e^{KH} & 2K^2e^{-KH} \\ 2k^3e^{kH} & -2k^3e^{-kH} & K(k^2+K^2)e^{KH} & -K(k^2+K^2)e^{-KH} \\ 2k^3 & -2k^3 & K(k^2+K^2) & -K(k^2+K^2) \end{vmatrix} , \tag{4-83}$$

$$T_{31}(k,s) = \begin{vmatrix} -ke^{kH} & ke^{-kH} & -Ke^{KH} & Ke^{-KH} \\ (k^2+K^2)e^{kH} & (k^2+K^2)e^{-kH} & 2K^2e^{KH} & 2K^2e^{-KH} \\ 2k^3e^{kH} & -2k^3e^{-kH} & K(k^2+K^2)e^{KH} & -K(k^2+K^2)e^{-KH} \\ 2k^3 & -2k^3 & K(k^2+K^2) & -K(k^2+K^2) \end{vmatrix} , \tag{4-84}$$

$$D_{31}(k,s) = \begin{vmatrix} -ke^{kH} & ke^{-kH} & -Ke^{KH} & Ke^{-KH} \\ -k & k & -K & K \\ 2k^3e^{kH} & -2k^3e^{-kH} & K(k^2+K^2)e^{KH} & -K(k^2+K^2)e^{-KH} \\ 2k^3 & -2k^3 & K(k^2+K^2) & -K(k^2+K^2) \end{vmatrix} , \tag{4-85}$$

$$\delta_{31}(k,s) = \begin{vmatrix} -k & k & -K & K \\ (k^2+K^2)e^{kH} & (k^2+K^2)e^{-kH} & 2K^2e^{KH} & 2K^2e^{-KH} \\ 2k^3e^{kH} & -2k^3e^{-kH} & K(k^2+K^2)e^{KH} & -K(k^2+K^2)e^{-KH} \\ 2k^3 & -2k^3 & K(k^2+K^2) & -K(k^2+K^2) \end{vmatrix} , \tag{4-86}$$

and

$$D_{21}(k,s) = \begin{vmatrix} -ke^{kH} & ke^{-kH} & -Ke^{KH} & Ke^{-KH} \\ k^2+K^2 & k^2+K^2 & 2K^2 & 2K^2 \\ 2k^3e^{kH} & -2k^3e^{-kH} & K(k^2+K^2)e^{KH} & -K(k^2+K^2)e^{-KH} \\ 2k^3 & -2k^3 & K(k^2+K^2) & -K(k^2+K^2) \end{vmatrix} . \tag{4-87}$$

A property of these six determinants to be used later is that they are all odd functions in the Fourier transform variable k, that is,

$$\delta_{11}(-k,s) = -\,\delta_{11}(k,s) \quad , \tag{4-88}$$

$$M_{31}(-k,s) = -\,M_{31}(k,s) \quad , \tag{4-89}$$

$$T_{31}(-k,s) = -\,T_{31}(k,s) \quad , \tag{4-90}$$

$$D_{31}(-k,s) = -\,D_{31}(k,s) \quad , \tag{4-91}$$

$$\delta_{31}(-k,s) = -\,\delta_{31}(k,s) \quad , \tag{4-92}$$

and

$$D_{21}(-k,s) = -\,D_{21}(k,s) \quad . \tag{4-93}$$

The space-time response of the displacement of the upper surface can now be computed with the inversion theorems for Laplace and complex Fourier transforms. We obtain directly the integral representation

$$\eta_1(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} dk\; e^{-ixk}\, \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} ds\; e^{ts}\, \eta_1(k,s) \tag{4-94}$$

where $\eta_1(k,s)$ is given by Eq. (4-81).

As an explicit example of this general result for arbitrary initial perturbations on both surfaces of the fluid sheet, consider initial cosine perturbations on each surface with a phase difference ε. That is, suppose that the initial displacements on the upper and lower surfaces are given by

$$\eta_1(x,o) = a_1 \cos(k_o x + \varepsilon) \tag{4-95}$$

and

$$\eta_2(x,o) = a_2 \cos(k_o x) \quad . \tag{4-96}$$

The complex Fourier transforms of these initial displacements are

$$\eta_1(k,o) = \pi\, a_1 \left[e^{i\varepsilon}\, \delta(k+k_o) + e^{-i\varepsilon}\, \delta(k-k_o)\right] \tag{4-97}$$

and

$$\eta_2(k,o) = \pi\, a_2 \left[\delta(k+k_o) + \delta(k-k_o)\right] \quad . \tag{4-98}$$

Upon combining Eqs. (4-81), (4-94), (4-97), and (4-98), carrying out the integration over the Fourier transform variable k, with the use of the shifting property of the Dirac delta function, and making use of the odd properties of the fourth-order determinants expressed in Eqs. (4-88)-(4-93), we find that the space-time response of the upper surface of the fluid sheet can be represented by the Bromwich integral

$$\eta_1(x,t) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \frac{ds\ e^{ts}}{\delta(k_o,s)} \Big\{ a_1 \cos(k_o x+\varepsilon) \ldots$$

$$\ldots \left[s\ \delta_{11}(k_o,s) + \frac{g^*}{\nu} M_{31}(k_o,s) \right]$$

$$- a_2 \cos(k_o x) \frac{g^*}{\nu} T_{31}(k_o,s) \Big\} \qquad (4\text{-}99)$$

in the complex s-plane. Here we have introduced the definition

$$\delta(k_o,s) = \delta_{11}(k_o,s)\ s^2 + \frac{sg^*}{\nu} \left[\delta_{31}(k_o,s) - D_{21}(k_o,s) \right] + \left(\frac{g^*}{\nu}\right)^2 D_{31}(k_o,s) \quad . \qquad (4\text{-}100)$$

4.3. Interface Motion between Two Half-Spaces of Incompressible, Viscous Fluids under Constant Acceleration

To determine the space-time response of the interface between two viscous fluids under constant acceleration we solve the two uncoupled partial differential equations of the stream function formulation in the linear approximation. In Region (1), defined by the inequalities $-\infty < x < \infty$ and $y > o$, we have

$$\rho_1 \frac{\partial}{\partial t} \nabla^2\ \psi_1 = \mu_1\ \nabla^2\ \nabla^2\ \psi_1 \quad , \qquad (4\text{-}101)$$

and in Region (2), $-\infty < x < \infty$ and $y < o$, we have

$$\rho_2 \frac{\partial}{\partial t} \nabla^2\ \psi_2 = \mu_2\ \nabla^2\ \nabla^2\ \psi_2 \quad . \qquad (4\text{-}102)$$

The interface kinematic condition is

$$\frac{\partial}{\partial t}\ \eta(x,t) = - \frac{\partial^2}{\partial x \partial y}\ \psi_1(x,o,t) \quad . \qquad (4\text{-}103)$$

The four required boundary conditions for this problem arise from the continuity of the τ_{yy}- and τ_{yx}-components of the stress tensor and the continuity of the x- and y-components of the velocity vector. By using pressure field expressions of the form given in Eq. (4-60) and stress tensor component expressions of the form given in Eq. (4-62) for each of the two regions, we find that the continuity of the τ_{yy}-component of the stress tensor on the interface leads to the boundary condition

$$\begin{aligned} &\rho_1 \frac{\partial^2}{\partial t \partial x}\ \psi_1(x,o,t) - \rho_1 g^*\ \eta(x,t) \\ &\quad - \mu_1 \frac{\partial^3}{\partial x^3}\ \psi_1(x,o,t) + \mu_1 \frac{\partial^3}{\partial x \partial y^2}\ \psi_1(x,o,t) \\ &\quad = \rho_2 \frac{\partial^2}{\partial t \partial x}\ \psi_2(x,o,t) - \rho_2\ g^*\ \eta(x,t) \\ &\quad - \mu_2 \frac{\partial^3}{\partial x^3}\ \psi_2(x,o,t) + \mu_2 \frac{\partial^3}{\partial x \partial y^2}\ \psi_2(x,o,t) \quad , \end{aligned} \qquad (4\text{-}104)$$

in the absence of surface tension. The continuity of the τ_{yx}-component of the stress tensor leads to the boundary condition

$$\mu_1 \left[\frac{\partial^3}{\partial y^3}\ \psi_1(x,o,t) - \frac{\partial^3}{\partial x^2 \partial y}\ \psi_1(x,o,t) \right] = \mu_2 \left[\frac{\partial^3}{\partial y^3}\ \psi_2(x,o,t) - \frac{\partial^3}{\partial x^2 \partial y}\ \psi_2(x,o,t) \right] \quad . \qquad (4\text{-}105)$$

The continuity of the x-component of the velocity vector requires that

$$\frac{\partial^2}{\partial y^2}\ \psi_1(x,o,t) = \frac{\partial^2}{\partial y^2}\ \psi_2(x,o,t) \quad , \qquad (4\text{-}106)$$

and that of the y-component of the velocity vector gives the boundary condition

$$\frac{\partial^2}{\partial x \partial y}\ \psi_1(x,o,t) = \frac{\partial^2}{\partial x \partial y}\ \psi_2(x,o,t) \quad . \qquad (4\text{-}107)$$

To solve the initial value, boundary value problem represented by Eqs. (4-101)-(4-107) we introduce the Laplace-Fourier transforms of the scalar functions ψ_i, defined by

$$\psi_i(k,y,s) = \int_0^\infty dt\ e^{-st} \ldots$$

$$\ldots \int_{-\infty}^{\infty} dx\ e^{ikx}\ \psi_i(x,y,t),\ (i=1,2)\ , \qquad (4\text{-}108)$$

and of the interface displacement by

$$\eta(k,s) = \int_0^\infty dt\ e^{-st} \int_{-\infty}^{\infty} dx\ e^{ikx}\ \eta(x,t)\ . \qquad (4\text{-}109)$$

For $y > o$, the Laplace-Fourier transform of Eq. (4-101) is

$$\frac{d^4}{dy^4}\psi_1(k,y,s) - \left(2k^2 + \frac{s}{\nu_1}\right)\frac{d^2}{dy^2}\psi_1(k,y,s)$$

$$+ k^2\left(k^2 + \frac{s}{\nu_1}\right)\psi_1(k,y,s) = 0\ , \qquad (4\text{-}110)$$

and for $y < o$, that of Eq. (4-102) is

$$\frac{d^4}{dy^4}\psi_2(k,y,s) - \left(2k^2 + \frac{s}{\nu_2}\right)\frac{d^2}{dy^2}\psi_2(k,y,s)$$

$$+ k^2\left(k^2 + \frac{s}{\nu_2}\right)\psi_2(k,y,s) = 0\ . \qquad (4\text{-}111)$$

To ensure bounded solutions at infinity we take the solution of Eq. (4-110) in the form

$$\psi_1(k,y,s) = A_1(k,s)\ e^{-ky} + A_2(k,s)\ e^{-K_1 y}\ , \qquad (4\text{-}112)$$

and that of Eq. (4-111) in the form

$$\psi_2(k,y,s) = B_1(k,s)\ e^{ky} + B_2(k,s)\ e^{K_2 y}\ , \qquad (4\text{-}113)$$

where

$$K_i^2 = k^2 + \frac{s}{\nu_i}\ ,\ (i=1,2)\ . \qquad (4\text{-}114)$$

The problem can now be reduced to a set of five inhomogeneous algebraic equations in which the unknowns are A_1 and A_2 in Eq. (4-112), B_1 and B_2 in Eq. (4-113), and $\eta(k,s)$, the Laplace-Fourier transform of the interface displacement. To accomplish this reduction we first take the Laplace-Fourier transforms of the kinematic and boundary conditions given in Eqs. (4-103)-(4-107). These five transforms read as follows:

$$s\ \eta(k,s) - \eta(k,o) = (ik)\frac{d}{dy}\psi_1(k,o,s)\ , \qquad (4\text{-}115)$$

$$\rho_1\ sik\ \psi_1(k,o,s) + \rho_1\ g^*\ \eta(k,s) + \mu_1\ ik^3\ \psi_1(k,o,s)$$

$$+ \mu_1\ ik\frac{\partial^2}{\partial y^2}\psi_1(k,o,s) = \rho_2\ sik\ \psi_2(k,o,s)$$

$$+ \rho_2\ g^*\ \eta(k,s) + \mu_2\ ik^3\ \psi_2(k,o,s)$$

$$+ \mu_2\ ik\frac{\partial^2}{\partial y^2}\psi_2(k,o,s)\ , \qquad (4\text{-}116)$$

$$\mu_1\left[\frac{\partial^3}{\partial y^3}\psi_1(k,o,s) + k^2\frac{\partial}{\partial y}\psi_1(k,o,s)\right]$$

$$= \mu_2\left[\frac{\partial^3}{\partial y^3}\psi_2(k,o,s) + k^2\frac{\partial}{\partial y}\psi_2(k,o,s)\right], \qquad (4\text{-}117)$$

$$\frac{\partial^2}{\partial y^2}\psi_1(k,o,s) = \frac{\partial^2}{\partial y^2}\psi_2(k,o,s)\ , \qquad (4\text{-}118)$$

and

$$\frac{\partial}{\partial y}\psi_1(k,o,s) = \frac{\partial}{\partial y}\psi_2(k,o,s)\ . \qquad (4\text{-}119)$$

Upon entering Eqs. (4-112) and (4-113) into Eqs. (4-116)-(4-119) we find that

$$\Delta_\nu \begin{pmatrix} A_1(k,s) \\ A_2(k,s) \\ \\ B_1(k,s) \\ B_2(k,s) \end{pmatrix} = \begin{pmatrix} \dfrac{(\rho_1 - \rho_2)}{(-ik)}\ g^*\ \eta(k,s) \\ 0 \\ \\ 0 \\ 0 \end{pmatrix}\ , \qquad (4\text{-}120)$$

where the matrix Δ_ν is given by

$$\Delta_\nu = \begin{bmatrix} \mu_1\left(k^2+K_1^2\right) & 2\mu_1 K_1^2 & -\mu_2\left(k^2+K_2^2\right) & -2\mu_2 K_2^2 \\ 2\mu_1 k^3 & \mu_1 K_1\left(k^2+K_1^2\right) & 2\mu_2 k^3 & \mu_2 K_2\left(k^2+K_2^2\right) \\ k & K_1 & k & K_2 \\ k^2 & K_1^2 & -k^2 & -K_2^2 \end{bmatrix} . \tag{4-121}$$

Also, Eq. (4-115) becomes

$$s\,\eta(k,s) - \eta(k,o) = (-ik)\ \left[kA_1(k,s) + K_1A_2(k,s)\right] . \tag{4-122}$$

Solving Eq. (4-120) for A_1 and A_2 yields

$$A_1(k,s) = \frac{\Delta_{\nu,13}\left(\rho_1 - \rho_2\right)}{\Delta_\nu\ (-ik)}\, g^*\ \eta(k,s) \tag{4-123}$$

and

$$A_2(k,s) = -\frac{\Delta_{\nu,23}\ \left(\rho_1 - \rho_2\right)}{\Delta_\nu\ (-ik)}\, g^*\ \eta(k,s) \quad , \tag{4-124}$$

where the two additional determinants are

$$\Delta_{\nu,13} = \begin{vmatrix} \mu_1\ K_1\left(k^2 + K_1^2\right) & 2\mu_2 k^3 & \mu_2 K_2\left(k^2+K_2^2\right) \\ K_1 & k & K_2 \\ K_1^2 & -k^2 & -K_2^2 \end{vmatrix} , \tag{4-125}$$

and

$$\Delta_{\nu,23} = \begin{vmatrix} 2\mu_1 k^3 & 2\mu_2 k^3 & \mu_2 K_2\left(k^2+K_2^2\right) \\ k & k & K_2 \\ k^2 & -k^2 & -K_2^2 \end{vmatrix} . \tag{4-126}$$

Combining Eqs. (4-122)-(4-124) produces

$$\eta(k,s) = \frac{\Delta_\nu\ \mu(k,o)}{s\Delta_\nu + \left(K_1\ \Delta_{\nu,23} - k\Delta_{\nu,13}\right)\ \left(\rho_1-\rho_2\right)g^*} \tag{4-127}$$

as the Laplace-Fourier transform of the interface displacement. A little algebra establishes the identity

$$K_1\ \Delta_{\nu,23} - k\ \Delta_{\nu,13} = s\ \Delta_{\nu,3} \quad , \tag{4-128}$$

where

$$\Delta_{\nu,3} = k^2\ K_1\ K_2\ \left[\rho_1\ K_2 + \rho_2\ K_1 - k\left(\rho_1+\rho_2\right)\right] . \tag{4-129}$$

Consequently, the Laplace-Fourier transform of the interface displacement becomes

$$\eta(k,s) = \frac{\mu(k,o)}{s} \frac{\Delta_\nu}{\Delta_\nu + \Delta_{\nu,3}\left(\rho_1-\rho_2\right)g^*} , \qquad (4\text{-}130)$$

and that of the time derivative of the interface displacement is

$$s\,\eta(k,s) - \eta(k,o) = \frac{-\eta(k,o)\,\Delta_{\nu,3}\left(\rho_1-\rho_2\right)g^*}{\Delta_\nu + \Delta_{\nu,3}\left(\rho_1-\rho_2\right)g^*} . \qquad (4\text{-}131)$$

By applying the inversion theorems for the Laplace and Fourier transforms the following integral representatives for the space-time response of the interface and its time derivative are obtained:

$$\eta(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} dk\; e^{-ixk}\,\eta(k,o) \ldots$$

$$\ldots \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \frac{ds}{s} \frac{\Delta_\nu\, e^{ts}}{\left[\Delta_\nu + \Delta_{\nu,3}\left(\rho_1-\rho_2\right)g^*\right]} \qquad (4\text{-}132)$$

and

$$\frac{\partial}{\partial t}\eta(x,t) = -\frac{1}{2\pi}\int_{-\infty}^{\infty} dk\; e^{-ixk}\,\eta(k,o) \ldots$$

$$\ldots \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} ds\, \frac{\left(\rho_1-\rho_2\right)g^*\Delta_{\nu,3}\; e^{ts}}{\left[\Delta_\nu + \Delta_{\nu,3}\left(\rho_1-\rho_2\right)g^*\right]} . \qquad (4\text{-}133)$$

For an initial cosine perturbation, for which

$$\eta(k,o) = \pi a_o \left[\delta\left(k+k_o\right) + \delta\left(k-k_o\right)\right] , \qquad (4\text{-}134)$$

we find that

$$\eta(x,t) = \frac{a_o \cos\left(k_o x\right)}{2\pi i} \ldots$$

$$\ldots \int_{c-i\infty}^{c+i\infty} \frac{ds}{s} \frac{\Delta_\nu\left(k_o,s\right)\, e^{ts}}{\left[\Delta_\nu\left(k_o,s\right) + \Delta_{\nu,3}\left(k_o,s\right)\left(\rho_1-\rho_2\right)g^*\right]}$$

and (4-135)

$$\frac{\partial}{\partial t}\eta(x,t) = -\frac{a_o \cos(k_o x)}{2\pi i} \ldots$$

$$\ldots \int_{c-i\infty}^{c+i\infty} ds\, \frac{\left(\rho_1-\rho_2\right)g^*\Delta_{\nu,3}\left(k_o,s\right)\, e^{ts}}{\left[\Delta_\nu\left(k_o,s\right) + \Delta_{\nu,3}\left(k_o,s\right)\left(\rho_1-\rho_2\right)g^*\right]} \qquad (4\text{-}136)$$

are the Bromwich integral representations of the interface displacement and its time derivative, respectively.

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